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Positive singular solutions of a certain elliptic PDE

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ABSTRACT

In this paper, we investigate the existence of positive singular solutions for a system of partial differential equations on a bounded domain

$$\begin{cases} -\Delta u = (1+\kappa_1(x))|\nabla v|^p & \text{in } B_1\backslash\{0\}, \\ -\Delta v = (1+\kappa_2(x))|\nabla u|^p & \text{in } B_1\backslash\{0\}, \\ u = v = 0 & \text{on } \partial B_1. \end{cases} \tag{1}$$

We investigate the existence of positive singular solutions within B_1, the unit ball centered at the origin in $\mathbb{R}^N$, under the conditions $N \geq 3$ and $\frac{N}{N-1} < p < 2$. Additionally, we assume that κ_1 and κ_2 are non-negative, continuous functions satisfying $\kappa_1(0) = \kappa_2(0) = 0$. Our system is an extension of the PDE studied by Aghajani et al. [1] under similar assumptions.

1. Introduction

Numerous studies have been done on the existence of positive singular solutions for various partial differential equations. One such example is the Lane-Emden equation.

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

For $p > 1$, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary. The existence and non-existence of solutions to this problem have been extensively studied on various bounded domains and for different ranges of p; see [14,8–10,7]. A closely related example to equation (1) is equation

$$\begin{cases} -\Delta w = |\nabla w|^p & \text{in } B_1\backslash\{0\}, \\ w = 0 & \text{on } \partial B_1. \end{cases} \tag{3}$$

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The case $0 < p < 1$ was analyzed in [2], while boundary blow-up versions of (3), where the negative sign in front of the Laplacian is removed, were studied in [12,15]. Additionally, various studies on equations similar to (3) can be found in [4,3,5,6,11].

In this work, we aim to establish the existence of positive singular solutions to equation (1) in the unit ball B_1, where $\frac{N}{N-1} < p < 2$ and B_1 is centered at the origin in $\mathbb{R}^N$ $(N \geq 3)$. Unlike previous studies, we impose no smallness conditions on κ_1 and κ_2 beyond the assumption that they vanish at the origin.

Our main result is as follows.

Theorem 1.1. *Suppose $N \geq 3$, $\frac{N}{N-1} < p < 2$ and κ_1, κ_2 are non-negative, continuous functions with $\kappa_1(0) = \kappa_2 = 0$. There exists an infinite number of positive singular solutions u_t, v_t of*

$$\begin{cases} -\Delta u = (1+\kappa_1(x))|\nabla v|^p & \text{in } B_1\backslash\{0\}, \\ -\Delta v = (1+\kappa_2(x))|\nabla u|^p & \text{in } B_1\backslash\{0\}, \\ u = v = 0 & \text{on } \partial B_1. \end{cases} \tag{4}$$

which blow up at the origin. Moreover, u_t and v_t converge uniformly to zero away from the origin as t approaches infinity.

Equation (4) is an extension of the equation studied by Aghajani et al. [1] where they analyzed the existence of positive singular solutions on bounded domains and also classical solutions on exterior domains. **The parameters.** The parameters p, ξ, β, and σ for the remainder of this work will be as follows unless otherwise stated.

First we consider the following singular solution of a scalar problem.

Example 1.2. Let $N \geq 3$, $\frac{N}{N-1} < p < 2$, $\xi := (p-1)(N-1)$ (note this implies $\xi > 1$), $\beta := \frac{p-1}{\xi-1} > 0$. Then for all $t \geq 0$ the radial function

$$w_t(r) = \int_r^1 \frac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}} \tag{5}$$

is the singular solution of

$$\begin{cases} -\Delta w = |\nabla w|^p & \text{in } B_1\backslash\{0\}, \\ w = 0 & \text{on } \partial B_1. \end{cases} \tag{6}$$

Remark 1.3. For the remaining sections of this work that focus on results in bounded domains, we adopt the parameter values from Example 1.2. This also applies to all the material presented in the Introduction.

Our approach for finding a solution (u, v) of (4) will be to look for solutions of the form $u = w_t + \varphi$ and $v = w_t + \psi$. Note that φ, and ψ are unknown functions, but they are equal to zero on the boundary of B_1, and w_t is the solution of (6). By considering our solutions u and v in this new form, φ and ψ need to satisfy

$$\begin{cases} -\Delta\varphi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi = \kappa_1(x)|\nabla w_t + \nabla\psi|^p + I(\psi) & \text{in } B_1\backslash\{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\varphi = \kappa_2(x)|\nabla w_t + \nabla\varphi|^p + I(\varphi) & \text{in } B_1\backslash\{0\}, \\ \varphi = \psi = 0 & \text{on } \partial B_1, \end{cases} \tag{7}$$

where

$$I(\zeta) := |\nabla w_t + \nabla\zeta|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\zeta.$$

To find a solution of (7), we will apply a fixed point argument. A key step will be to understand the linear operator on the left hand side of (7), namely the solvability of

$$\begin{cases} -\Delta\varphi - p|\nabla w_t|^{p-2}\nabla w_t\cdot\nabla\psi = f & \text{in } \ B_1\backslash\{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t\cdot\nabla\varphi = g & \text{in } \ B_1\backslash\{0\}, \\ \varphi=\psi=0 & \text{on } \ \partial B_1, \end{cases} \tag{8}$$

for (φ,ψ), given f and g.

We define the nonlinear mapping $T_t(\varphi,\psi)=(\hat{\varphi},\hat{\psi})$ via

$$\begin{cases} -\Delta\hat{\varphi} - p|\nabla w_t|^{p-2}\nabla w_t\cdot\nabla\hat{\psi} = \kappa_1(x)|\nabla w_t+\nabla\psi|^p + I(\psi) & \text{in } \ B_1\backslash\{0\}, \\ -\Delta\hat{\psi} - p|\nabla w_t|^{p-2}\nabla w_t\cdot\nabla\hat{\varphi} = \kappa_2(x)|\nabla w_t+\nabla\varphi|^p + I(\varphi) & \text{in } \ B_1\backslash\{0\}, \\ \varphi=\psi=0 & \text{on } \ \partial B_1. \end{cases} \tag{9}$$

In section 3, will show that T_t is a contraction on a suitable complete metric space. Subsequently, we will use this result to prove the existence of the positive singular solutions of (4). We fix (f,g), and set $F\coloneqq f+g$ and $G\coloneqq f-g$. Suppose that ζ_i is a solution of the scalar problems

$$\begin{cases} -\Delta\zeta_1 - p|\nabla w_t|^{p-2}\nabla w_t\cdot\nabla\zeta_1 = F & \text{in } \ B_1\backslash\{0\}, \\ \zeta_1=0 & \text{on } \ \partial B_1, \end{cases} \tag{10}$$

$$\begin{cases} -\Delta\zeta_2 + p|\nabla w_t|^{p-2}\nabla w_t\cdot\nabla\zeta_2 = G & \text{in } \ B_1\backslash\{0\}, \\ \zeta_2=0 & \text{on } \ \partial B_1. \end{cases} \tag{11}$$

The problem in (10) has been studied in the previous work [1]. In section 2, we will prove the results for (11). A linear algebra argument shows that if ζ_i satisfies the above, then for (φ,ψ) to be a solution of (8), we need φ,ψ to satisfy $\varphi+\psi=\zeta_1$, and $\varphi-\psi=\zeta_2$. From this, we see that

$$\varphi=\frac{\zeta_1+\zeta_2}{2} \quad \text{and} \quad \psi=\frac{\zeta_1-\zeta_2}{2}. \tag{12}$$

So to understand the solvability of (8), it is enough to understand the solvability of the two scalar problems given by (10) and (11). One can write out the left hand sides of (10) and (11) explicitly respectively as

$$-\Delta\zeta_1+\frac{p}{(t|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\zeta_1}{|x|^2}, \quad -\Delta\zeta_2-\frac{p}{(t|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\zeta_2}{|x|^2}.$$

This motivates the definition of the linear operators

$$L_t^{\pm}(\zeta)=-\Delta\zeta\pm\frac{p}{(t|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\zeta}{|x|^2}. \tag{13}$$

So L_t^+ is the linear operator associated with the left hand side of (10) and L_t^- is the linear operator associated with the left-hand side of (11). Later we will need the following asymptotic result:

- $\lim_{r\to 0} r^{\sigma+1}w_t'(r)=-C_\beta \quad \text{where} \quad C_\beta=\dfrac{1}{\beta^{\frac{1}{p-1}}}$ (14)
- for all $t\geq 0$, there exists a constant C such that $\lim_{r\searrow 0} r^{\sigma}w_t(r)=C$.

2. The linear theory

2.1. Analysis of the linear operators $L_t^{\pm}$

Define the following norms

$$\|\varphi\|_X := \sup_{0<|x|\le 1}\{|x|^{\sigma}|\varphi| + |x|^{\sigma+1}|\nabla\varphi|\}, \quad \|f\|_Y := \sup_{0<|x|\le 1}\{|x|^{\sigma+2}|f(x)|\}$$

where $\sigma = \frac{2-p}{p-1}$. Let X denote the set of functions φ such that $\|\varphi\|_X < \infty$ and vanish on the boundary of B_1 and let Y denote the set of functions f such that $\|f\|_Y < \infty$. The goal is to show that our nonlinear mapping $T_t(\varphi,\psi)$ is a contraction by applying Banach's Fixed Point Theorem on the complete metric space

$$\mathcal{F}_R := \{(\varphi,\psi) \in X \times X : \|\varphi\|_X, \|\psi\|_X \le R\}.$$

On this space, we have

$$\|(\varphi,\psi)\|_{X\times X} := \|\varphi\|_X + \|\psi\|_X.$$

We can now state our main proposition in this chapter.

Proposition 2.1. *Let $N \ge 3$, such that $\frac{N}{N-1} < p < 2$, $\xi = (p-1)(N-1) > 1$, $\beta = \frac{p-1}{\xi-1} > 0$ and $\sigma = \frac{2-p}{p-1}$. There is a positive constant C such that for all $f,g \in Y$, there are functions φ and ψ in X which satisfy the equation*

$$\begin{cases} -\Delta\varphi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi = f & \text{in } B_1\backslash\{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\varphi = g & \text{in } B_1\backslash\{0\}, \\ \varphi = \psi = 0 & \text{on } \partial B_1, \end{cases} \tag{15}$$

and the estimate

$$\|\varphi\|_X + \|\psi\|_X \le C\|f\|_Y + C\|g\|_Y.$$

With the approach we have taken in this section, we use the change of notation from (12) and restate our proposition as follows.

Proposition 2.2. *Let $N \ge 3$, such that $\frac{N}{N-1} < p < 2$, $\xi = (p-1)(N-1) > 1$, $\beta = \frac{p-1}{\xi-1} > 0$ and $\sigma = \frac{2-p}{p-1}$. There is some positive constant C such that for all $f \in Y$ and non-negative t, there is some $\varphi \in X$ which satisfies*

$$\begin{cases} L_t^{\pm}(\varphi) = f & \text{in } B_1\backslash\{0\}, \\ \varphi = 0 & \text{on } \partial B_1. \end{cases} \tag{16}$$

Moreover, one has the estimate $\|\varphi\|_X \le C\|f\|_Y$.

Spherical harmonics. Note $\Delta_\theta = \Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1} with eigenpairs (ψ_k, λ_k) satisfying

$$-\Delta_\theta\psi_k(\theta) = \lambda_k\psi_k(\theta) \quad \text{for} \quad \theta \in S^{N-1}$$

and $\psi_0 = 1$, $\lambda_0 = 0, \lambda_1 = N-1, \lambda_2 = 2N$. We normalize ψ_k in $L^2(S^{N-1})$ such that $\|\psi_k\|_{L^2(S^{N-1})} = 1$.

Using this, given $f \in Y$, and $\varphi \in X$, we can decompose φ and f into various modes by writing

$$f(x) = \sum_{k=0}^{\infty} b_k(r)\psi_k(\theta) \qquad \text{and} \qquad \varphi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta).$$

Note that $a_k(1) = 0$ after considering the boundary condition of φ. A computation shows that φ satisfies (16) provided a_k satisfies

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + \frac{\lambda_k a_k(r)}{r^2} \pm \frac{pa_k'(r)}{\beta r + tr^{\xi}} = b_k(r) \quad \text{for} \;\; 0 < r < 1 \tag{17}$$

with $a_k(1) = 0$.

Let X_1 and Y_1 be the closed subspaces of X and Y defined by

$$X_1 := \left\{ \varphi \in X : \varphi = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta) \right\}, \qquad Y_1 := \left\{ f \in Y : f = \sum_{k=1}^{\infty} b_k(r)\psi_k(\theta) \right\}.$$

Note that X_1 and Y_1 are just the same representations as X and Y except that the summations start at $k = 1$ and not $k = 0$.

To show that they are closed subspaces of X and Y, we need to show that if any sequence $\varphi_m \in X_1$ is such that φ_m converges to some φ in X, then φ is in X_1. We need a similar result for Y and Y_1. We first note that if we have $\varphi(x) \in X_1$ then we can write $\varphi(r,\theta) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$ and by integrating both sides with respect to θ, we get

$$\int_{S^{N-1}} \varphi(r,\theta) d\theta = \sum_{k=1}^{\infty} a_k(r) \int_{S^{N-1}} \psi_k(\theta) d\theta = 0.$$

This is due to the orthonormality of ${\psi_k}'s$ and $\psi_0 = 1$. Thus, we can conclude that for all φ in X we have

$$\varphi \in X_1 \Longleftrightarrow \int_{S^{N-1}} \varphi(r,\theta) = 0 \quad \text{for all} \;\; 0 < r \le 1. \tag{18}$$

Since by the assumption we have $\varphi_m(r,\theta) \in X_1$, it follows that $\int_{S^{N-1}} \varphi_m(r,\theta) = 0$ for all $0 < r \le 1$. Fix $0 < r \le 1$ and set $\zeta_m(\theta) = \varphi_m(r,\theta)$ and $\zeta(\theta) = \varphi(r,\theta)$. We claim that ζ_m converges uniformly to ζ in X on S^{N-1}. Assuming the claim is true, we can fix $0 < r \le 1$ and by the uniform convergence of ζ_m to ζ on S^{N-1} we get

$$0 = \int_{S^{N-1}} \varphi_m(r,\theta) \longrightarrow \int_{S^{N-1}} \varphi(r,\theta).$$

Thus we have $\int_{S^{N-1}} \varphi(r,\theta) = 0$ and by (18) we can see that $\varphi \in X_1$. We now prove the claim. If we fix $\epsilon > 0$, we can write

$$\|\varphi_m - \varphi\|_X \ge \sup_{0<|x|\le 1} |x|^{\sigma} |\varphi_m(x,\theta) - \varphi(x,\theta)| \ge \sup_{\epsilon<|x|\le 1} |x|^{\sigma} |\varphi_m(x,\theta) - \varphi(x,\theta)| \ge \epsilon^{\sigma} |\varphi_m(x,\theta) - \varphi(x,\theta)|.$$

Note that $r = |x|$ and ϵ are fixed and we know that since φ_m converges to φ in X we have $\|\varphi_m - \varphi\|_X \to 0$. So we get the uniform convergence of φ_m to φ away from the origin. We can use a similar proof for Y_1. Thus X_1, and Y_1 are closed subspaces of X and Y respectively.

2.2. Kernel of $L_0^{\pm}$

Using the definition of our linear operator (13) in the case of $t = 0$, we can write it explicitly as

$$L_0^{\kappa}(\varphi) = -\Delta\varphi + \frac{\kappa x \cdot \nabla\varphi(x)}{\beta|x|^2}$$

where κ could be either equal to $+p$ or $-p$. For simplicity, we set $L_0^{+p} := L_0^{+}$ and $L_0^{-p} := L_0^{-}$.

Lemma 2.3. *Let $\frac{N}{N-1} < p < 2$, and we have the parameters*

$$\xi = (p-1)(N-1) > 1, \quad \beta = \frac{p-1}{\xi - 1} > 0, \quad \sigma = \frac{2-p}{p-1}.$$

Suppose $\varphi \in X_1$ satisfies $L_0^{\kappa}(\varphi) = 0$ in $B_R \setminus \{0\}$, with $\varphi = 0$ on ∂B_R when $R < \infty$. If $\kappa = p$ or $\kappa = -p$, then $\varphi = 0$.

Proof. We saw that we can decompose $\varphi(x)$ into $\varphi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$, and since $L_0^{\kappa} = 0$ in $B_R \setminus\{0\}$, then a_k should satisfy

$$-a_k''(r) - \frac{(N-1)a_k'}{r} + \frac{\lambda_k a_k(r)}{r^2} + \frac{\kappa a_k'(r)}{\beta r} = 0 \quad \text{for} \quad 0 < r < R \tag{19}$$

where $a_k(R) = 0$ in the case of $R < \infty$. Also we have

$$\sup_{0<r<R} \{r^{\sigma}|a_k(r)| + r^{\sigma+1}|a_k'(r)|\} < \infty. \tag{20}$$

Proof of the statement. To show that (20) is true, note that given $\varphi \in X$, we have $\varphi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta)$ with $a_k(R) = 0$. By using the property of orthonormality of $\psi_k(\theta)'s$, we get

$$a_k(r) = \int\limits_{S^{N-1}} \varphi(r,\theta)\psi_k(\theta)d\theta.$$

Thus, noting that $|\varphi(r,\theta)|r^{\sigma} \le \|\varphi\|_X$, we obtain

$$|a_k(r)| \le \sup_{S^{N-1}} |\psi_k(\theta)| \int\limits_{|\theta|=1} \frac{\|\varphi\|_X}{r^{\sigma}} d\theta.$$

Since φ is in X, there exists a positive C_k such that $\|\varphi\|_X \le C$. Thus we get $|a_k(r)| \le \frac{C_k}{r^{\sigma}}$ and we deduce

$$r^{\sigma}|a_k(r)| \le C_k. \tag{21}$$

Using the definition of the gradient in n-dimensional spherical coordinates and the orthogonal properties, we obtain $\varphi_r = \nabla\varphi(x) \cdot \hat{r}$ which gives us $|\varphi_r| \le |\nabla\varphi|$. Again for $\varphi \in X$ we can write $\varphi_r(r,\theta) = \sum_{k=0}^{\infty} a_k'(r)\psi_k(\theta)$ where $a_k(1) = 0$. We use the orthonormality of $\psi_k(\theta)'s$, and we obtain

$$a_k'(r) = \int\limits_{S^{N-1}} \varphi_r(r,\theta)\psi_k(\theta)d\theta.$$

Thus, noting that we have $|\nabla\varphi(r,\theta)|r^{\sigma+1} \le \|\varphi\|_X$, we can write

$$|a_k'(r)| \leq \sup_{|\theta|=1} |\psi_k(\theta)| \int_{S^{N-1}} |\varphi_r(r,\theta)| d\theta \leq \hat{C}_k \int_{|\theta|=1} |\nabla\varphi(r\theta)| d\theta \leq \hat{C}_k \int_{|\theta|=1} \frac{\|\varphi\|_X}{r^{\sigma+1}} \leq \frac{\tilde{C}_k}{r^{\sigma+1}}.$$

From this, we deduce

$$r^{\sigma+1}|a_k'(r)| < \tilde{C}. \tag{22}$$

The results from (21) and (22) give us $\sup_{0<r<R} \{r^\sigma |a_k(r)| + r^{\sigma+1}|a_k'(r)\} < \infty$. □

Note that the equation (19) is an Euler ODE. We rewrite it as

$$r^2 a_k''(r) + r a_k'(r) \left((N-1) - \frac{\kappa}{\beta} \right) - \lambda_k a_k(r) = 0$$

and its solution can be written as $a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$ where $\gamma^\pm$ are given by

$$\gamma_k^\pm(\kappa) = \frac{-(N-2-\frac{\kappa}{\beta})}{2} \pm \frac{\sqrt{(N-2-\frac{\kappa}{\beta})^2 + 4\lambda_k}}{2}. \tag{23}$$

First, we let $\kappa = -p$.

We will be looking at the kernel of $L_0^- = L_0^{-p}$. Thus our ODE becomes

$$-a_k''(r) - \frac{(N-1)a_k'}{r} + \frac{\lambda_k a_k(r)}{r^2} - \frac{p a_k'(r)}{\beta r} = 0 \quad \text{for} \quad 0 < r < R \tag{24}$$

and the solution can be written as $a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$ for some $C_k, D_k \in \mathbb{R}$ where $\gamma^\pm$ are given by (23) where $\kappa = -p$. We claim that $\gamma_k^- + \sigma \leq -1$ for $k \geq 1$. To show this, consider the function $f(\gamma) = \gamma^2 + (N-2+\frac{p}{\beta})\gamma - \lambda_1$. Note that $-\sigma - 1 = \frac{-1}{p-1}$, and using the definitions of β and ξ, we compute $f(-\sigma-1) = \frac{2p(1-\xi)}{(p-1)^2}$. Since $\xi > 1$, it follows that $f(-\sigma-1) < 0$. The function $f(\gamma)$ is a quadratic with roots $\gamma_1^\pm$, so the inequality $f(-\sigma-1) < 0$ implies that $\gamma_1^- \leq -\sigma - 1 \leq \gamma_1^+$. By monotonicity of γ_k^- with respect to k, it follows that $\gamma_k^- \leq -\sigma - 1$ for all $k \geq 1$. We now show that $\gamma_k^+ + \sigma$ is nonzero and positive. Note that we have

$$\gamma_1^+(-p) = \frac{-(N-2+\frac{p}{\beta})}{2} + \frac{\sqrt{(N-2+\frac{p}{\beta})^2 + 4(N-1)}}{2} > 0.$$

Thus by the monotonicity in k, we see that $\gamma_k^+ + \sigma$ is positive for all $k \geq 1$.

We first consider the case where $0 < R < \infty$. By considering the boundary condition, to have $a_k(r)$ satisfy (20), we must have $a_k(r) = 0$. For the case $R = \infty$, we note that $\gamma_k^- + \sigma$ is negative, and $\gamma_k^+ + \sigma$ is positive and they are distinct. By sending r to zero and infinity, we deduce that in order to have a_k satisfy (20), we must have $a_k = 0$. This shows that for all $k \geq 1$

$$\varphi = 0. \tag{25}$$

The proof will be very similar for the case $\kappa = p$ and a simplified proof of this case can also be found in [1] and we skip the proof here. Thus we proved the lemma, and so we showed that for $\varphi \in X_1$ if $L_0^\kappa(\varphi) = 0$ in $B_R \backslash \{0\}$ with $\varphi = 0$ on ∂B_R in the case of R finite where $\kappa = p$ or $\kappa = -p$, then $\varphi = 0$. □

2.3. Kernel of $L_t^{\pm}$

Recall that we have our linear operator L_t^k defined as:

$$L_t^{\kappa}(\varphi) = -\Delta\varphi \pm \frac{p}{(t|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\varphi}{|x|^2} \tag{26}$$

where κ is either equal to p or $-p$.

Lemma 2.4. *Let $0 < R \le \infty$ and $0 \le t \le \infty$.*

- *If $\kappa = p$, suppose $\varphi \in X_1$ satisfies*

$$L_t^p(\varphi) = 0 \quad \text{in } B_R \setminus \{0\}, \quad \text{with } \varphi = 0 \text{ on } \partial B_R \text{ (if } R \text{ is finite)},$$

then $\varphi = 0$.

- *If $\kappa = -p$, suppose $\varphi \in X$ satisfies*

$$L_t^{-p}(\varphi) = 0 \quad \text{in } B_R \setminus \{0\}, \quad \text{with } \varphi = 0 \text{ on } \partial B_R \text{ (if } R \text{ is finite)},$$

then in the case where R is finite, we have $\varphi = 0$, and if $R = \infty$, then φ must be constant.

Proof. For the proof we will switch notations, and hence by (26) we can write $L_t^+ = L_t^p$ and $L_t^- = L_t^{-p}$. Suppose R, t, φ are as in the hypothesis.
• We set $\kappa = +p$. So we are considering the kernel of L_t^+. A very similar proof was given in [1] so we will skip it here and we will do the case $\kappa = -p$ which is more technical. □

• We now set $\kappa = -p$. In this part, we focus on kernel of L_t^-. First we set $k \ge 1$. Thus φ is in X_1. Suppose $0 < t < \infty$. We write φ as $\varphi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$, and then a_k should satisfy

$$-a_k''(r) - \frac{(N-1)a_k'}{r} + \frac{\lambda_k a_k(r)}{r^2} - \frac{p a_k'(r)}{\beta r + t r^{\xi}} = 0 \quad \text{for} \quad 0 < r < R$$

where $a_k(R) = 0$ when $R < \infty$. We should have

$$\sup_{0<r<R}\{r^{\sigma}|a_k(r)| + r^{\sigma+1}|a_k'(r)|\} < \infty. \tag{27}$$

We fix $k \ge 1$ and we set $\omega(\tau) = r^{\sigma}a_k(r)$ where $\tau = \ln(r)$. By a computation, we find that $\omega = \omega(\tau)$ satisfies $\omega_{\tau\tau} + g(\tau)\omega_\tau + C_k\omega = 0$ for $\tau \in (-\infty, \ln(R))$ where $g(\tau) = N - 2 - 2\sigma + \frac{p}{\beta + te^{(\xi-1)\tau}}$ and $C_k(\tau) = -\lambda_k - \frac{p\sigma}{\beta+te^{(\xi-1)\tau}} - \sigma(N-2-\sigma)$. We claim the improved decay estimate: $r^{\sigma}|a_k(r)| \to 0$ as $r \to 0$, and in the case $R = \infty$, $r^{\sigma}|a_k(r)| \to 0$ as $r \to \infty$. Assuming this holds, it follows that $\omega \to 0$ as $\tau \to -\infty$, and for $R = \infty$, also as $\tau \to \infty$. By multiplying by -1 if necessary, we assume $\omega \ne 0$. Since $\omega(-\infty) = \omega(\ln R) = 0$, there exists some $\tau_0 \in (-\infty, \ln R)$ such that $\omega(\tau_0) = \max\omega > 0$. This gives $\omega_{\tau\tau}(\tau_0) \le 0$ and $\omega_\tau(\tau_0) = 0$, leading to $g(\tau_0)\omega_\tau(\tau_0) = 0$. From the equation, we obtain $\omega_{\tau\tau}(\tau_0) + C_k\omega(\tau_0) = 0$ which implies $-\omega_{\tau\tau}(\tau_0) = C_k\omega(\tau_0) \ge 0$. From this, we see that we must have

$$C_k(\tau_0) = -\lambda_k - \frac{p\sigma}{\beta + te^{(\xi-1)\tau_0}} - \sigma(N-2-\sigma) \ge 0.$$

We know λ_k is positive for all $k \ge 1$ and it is obvious that $\frac{p\sigma}{\beta+te^{(\xi-1)\tau}} > 0$ for all $\tau \in (-\infty, \ln(R))$. Also note that $N - 2 - \sigma = N - 1 - \frac{1}{p-1}$. By considering the restrictions on p, we find that $N - 1 - \frac{1}{p-1} > 0$ for $N \ge 3$. Thus we can deduce that

$$-\lambda_k < 0, \quad -\frac{p\sigma}{\beta + te^{(\xi-1)\tau}} < 0, \quad \text{and} \quad -\sigma(N-2-\sigma) < 0.$$

So $C_k(\tau)$ is positive for all $\tau \in (-\infty, \ln(R))$ which means $C_k(\tau_0) < 0$. Hence, we have a contradiction and thus $\omega = 0$. This gives us $a_k = 0$ for all $k \geq 1$.

We now prove the claimed decay estimate.

Proof of the claimed decay estimate. We fix $k \geq 1$ and set $a(r) = a_k(r)$, so we have

$$-\Delta a(r) + \frac{\lambda_k a(r)}{r^2} - \frac{pa'(r)}{\beta r + tr^\xi} = 0 \quad \text{in} \quad 0 < r < R,$$

with $a(R) = 0$. Suppose the claim is false. Then there is some r_m that goes to zero such that $r_m^\sigma |a(r_m)| \geq \epsilon_0 > 0$. Define the rescaled function $a^m(r) := r_m^\sigma a(r_m r)$ and note that $|a_m(1)| \geq \epsilon_0$ and $r^\sigma |a^m(r)| \leq C$. A computation shows that

$$r_m{}^{\sigma+2}\left[\Delta a(r_m r) + \frac{\lambda_k a(r_m r)}{(r_m r)^2} - \frac{pa'(r_m r)}{\beta r_m r + t(r_m r)^\xi}\right] = 0 \qquad \text{in} \;\; 0 < r_m r < R,$$

so we get

$$-\Delta a^m(r) + \frac{\lambda_k a^m(r)}{r^2} - \frac{p(a^m)'(r)}{\beta r + tr^\xi r_m^{\xi-1}} = 0 \quad \text{in} \quad 0 < r < \frac{R}{r_m}. \tag{28}$$

By passing to the limit, we find a^∞ such that it is bounded away from zero with $r^\sigma |a^\infty(r)| + r^{\sigma+1}|(a^\infty)'(r)| \leq C$. Thus, we have

$$-\Delta a^\infty(r) + \frac{\lambda_k a^\infty(r)}{r^2} + \frac{p(a^\infty)'(r)}{\beta r} = 0 \quad \text{in} \quad 0 < r < \infty.$$

Let $\varphi(x) = a^\infty(r)\psi_k(\theta)$. So φ is nonzero too and it is in the kernel of L_0^-. This is a contradiction with Lemma 2.3 which stated the kernel of L_0^- is trivial.

In the case of $R = \infty$, we assume there exists some r_m approaching infinity as m goes to infinity. Again we pass to the limit in (28) and we get

$$-(a^\infty)''(r) - \frac{(N-1)(a^\infty)'}{r} + \frac{\lambda_k a^\infty(r)}{r^2} = 0 \quad \text{in} \quad 0 < r < \infty.$$

This is again an Euler ODE and the characteristic equation would be $\gamma^2 + (N-2)\gamma - \lambda_k = 0$ such that

$$\gamma_k^\pm = \frac{-(N-2) \pm \sqrt{(N-2)^2 + 4\lambda_k}}{2}.$$

Thus, the solution is $a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$. We have that $\gamma_k^- + \sigma < 0 < \gamma_k^+ + \sigma$. So we can deduce that when we send r to zero and infinity, we must have $C_k = D_k = 0$ in order to have a_k in the required space. This result gives us that $a_k = 0$ for all $k \geq 1$.

• We now set $k = 0$. We have $L_t^- = 0$ thus we can write

$$-a_k''(r) - \frac{(N-1)a_k'}{r} - \frac{pa_k'(r)}{\beta r + tr^\xi} = 0 \quad \text{for} \quad 0 < r < R \tag{29}$$

such that $\sup\limits_{0<r<1} \{r^\sigma |a_t(r)| + r^{\sigma+1}|a_t'(r)|\} \leq C$. From (29), using the integrating factor $\mu_t(r)$, we get:

$$\frac{d}{dr}(\mu_t(r)a_t'(r)) = 0 \Rightarrow \mu_t a'(r) = C$$

where C is a constant. Thus we have

$$r^{\sigma+1}|a_t'(r)| = \frac{Cr^{\sigma+1}}{\mu_t(r)} = Cr^{\sigma+1-N+1-\frac{p}{\beta}}\left(\frac{tr^{\xi-1}+\beta}{t+\beta}\right)^{\frac{p}{p-1}}.$$

Note that $\sigma+2-N-\frac{p}{\beta} = \frac{(\xi-1)(-1-p)}{p-1} < 0$. Since φ is in X, we require a_t' to satisfy the required bounds meaning $r^{\sigma+1}|a_t'(r)|$ should be bounded. So we can deduce that C should be zero. So we have

$$\mu_t a_t'(r) = 0.$$

By considering the boundary condition $a_t(R) = 0$, we obtain $a_t(r) = 0$ for $r \in (0, R)$ and $a_t(r)$ is constant on the whole space $\mathbb{R}^N$. We now consider the case where t approaches infinity. Thus we have the ODE

$$-a_k''(r) - \frac{(N-1)a_k'}{r} = 0 \tag{30}$$

where $a(1) = 0$. We have the integrating factor as $\mu_t(r) = r^{(N-1)}$, thus, we have $\frac{d}{d\tau}(\mu_t(\tau)a_t'(\tau)) = 0$. Now by solving this equation, we obtain

$$a(r) = C\int_r^1 \frac{1}{s^{N-1}}ds = \frac{C}{2-N}s^{2-N}\Big|_r^1 = C\left(\frac{1}{2-N} - \frac{r^{2-N}}{2-N}\right).$$

We know $a(r)$ needs to satisfy $r^\sigma|a_t(r)| \le C_1$ for some positive constant C_1. So we should have

$$r^\sigma\left|C\left(\frac{1}{2-N} - \frac{r^{2-N}}{2-N}\right)\right| = \left|C\left(\frac{r^\sigma}{2-N} - \frac{r^{\sigma+2-N}}{2-N}\right)\right| \le C_1.$$

Noting that $\sigma = \frac{2-p}{p-1} > 0$ and $\frac{N}{N-1} < p < 2$, by a computation we can see that $\sigma+2-N$ is negative. So for $a_t(r)$ to satisfy the required bound when r approaches zero, we should have $C = 0$. This shows that $a_t(r) = 0$ when t goes to infinity and so we should have $\varphi = 0$.

Thus the lemma is complete and we proved if $\kappa = p$ and $\varphi \in X_1$ is such that $L_t^p(\varphi) = 0$ in $B_R\backslash\{0\}$ with $\varphi = 0$ on ∂B_R in the case of R finite then $\varphi = 0$ and when $\kappa = -p$ and $\varphi \in X$ is such that $L_t^{-p}(\varphi) = 0$ in $B_R\backslash\{0\}$ with $\varphi = 0$ on ∂B_R in the case of R finite then $\varphi = 0$ and in case of $k = 0$ and R infinite φ is a constant. □

2.4. Linear theory of $L_t^\pm$ on X_1; a priori estimate

Theorem 2.5. *There is some positive constant C such that for all positive t_m, and functions $\varphi_m \in X_1$ and $f^m \in Y_1$ we have*

$$\begin{cases} L_{t_m}^\pm(\varphi_m) = f^m & in \ \ B_1\backslash\{0\}, \\ \varphi_m = 0 & on \ \ \partial B_1. \end{cases} \tag{31}$$

One has the estimate $\|\varphi_m\|_X \le C\|f^m\|_Y$.

Proof. If we assume the result is false, then by passing to a subsequence (without renaming) there is some $C_m > 0$, $t_m \ge 0$, $f^m \in Y_1$ and $\varphi_m \in X_1$ with

$$\begin{cases} -\Delta\varphi_m + \dfrac{\kappa}{(t_m|x|^{\xi-1}+\beta)}\dfrac{x\cdot\nabla\varphi_m}{|x|^2} = f^m & \text{in } \; B_1\backslash\{0\}, \\ \varphi_m = 0, & \text{on } \; \partial B_1, \end{cases} \tag{32}$$

and

$$\|\varphi_m\|_X > C_m\|f^m\|_Y.$$

By normalizing, we get $\|\varphi_m\|_X = 1$ and $\|f^m\|_Y \to 0$.

We claim

$$\sup_{0<|x|\le 1}\{|x|^{\sigma+1}|\nabla\varphi_m|\} \to 0. \tag{33}$$

We will show that proving this claim results in: $\sup_{0<|x|\le 1}\{|x|^{\sigma}|\varphi_m|\} \to 0$.

Proof of the claim. Suppose there is some $0 < |x_m| < 1$ and $\epsilon_0 > 0$ such that

$$\epsilon_0 \le |x_m|^{\sigma+1}|\nabla\varphi_m(x_m)| \le 1. \tag{34}$$

There are two cases that should be considered. Either $|x_m|$ could be bounded away from zero or it could be approaching zero.

• Set $\kappa = p$. Now we prove the first case for $L^+_{t_m}$.

Case 1: Assume $|x_m|$ is bounded away from zero. Define A_k and $\tilde{A}_k$ for $k \ge 2$ as

$$A_k = \Big\{x \in B_1 : \frac{1}{k} < |x| < 1\Big\} \quad \text{and} \quad \tilde{A}_k = \Big\{x \in B_1 : \frac{1}{2k} < |x| < 1\Big\}.$$

Note that $A_k \subset \tilde{A}_k$. We have

$$-\Delta\varphi_m = f^m - \frac{p}{(t_m|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\varphi_m}{|x|^2} \quad \text{in } \; \tilde{A}_k.$$

Set $g^m := f^m - \dfrac{p}{(t_m|x|^{\xi-1}+\beta)}\dfrac{x\cdot\nabla\varphi_m}{|x|^2}$. We can see that

$$\begin{aligned} \Big|\frac{p}{(t_m|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\varphi_m}{|x|^2}\Big| &\le \frac{p}{(t_m|x|^{\xi-1}+\beta)}\Big|\frac{x\cdot\nabla\varphi_m}{|x|^2}\Big| \\ &\le \frac{p}{\beta}\frac{\|\varphi_m\|_X}{|x|^{\sigma+2}} = \frac{p}{\beta}\frac{1}{|x|^{\sigma+2}} \le \frac{p}{\beta}(2k)^{2+\sigma} := \gamma_k. \end{aligned} \tag{35}$$

Also we find that,

$$\sup_{x\in\tilde{A}_k}|f^m| = \sup_{x\in\tilde{A}_k}\frac{|x|^{\sigma+2}|f^m|}{|x|^{\sigma+2}} \le \Big(\sup_{x\in\tilde{A}_k}\frac{\|f^m\|_Y}{|x|^{\sigma+2}}\Big) \le (2k)^{2+\sigma}\|f^m\|_Y = \gamma_k\|f^m\|_Y \tag{36}$$

which by (35) and (36) we can see that g_m is bounded in $\tilde{A}_k$. So there exists a positive constant C such that

$$\|g^m(x)\|_{L^\infty(\tilde{A}_k)} \le C$$

meaning

$$\|\Delta\varphi_m\|_{L^\infty(\tilde{A}_k)} \leq C.$$

By elliptic regularity, we can say that for some $0 < \lambda < 1$, there exists $C_1 > 0$ such that

$$\|\varphi_m\|_{C^{1,\lambda}(\bar{A}_k)} \leq C_1.$$

Thus $\{\varphi_m\}_m$ is a bounded sequence in $C^{1,\lambda}(\bar{A}_k)$ for all $k \geq 2$. By the standard compactness argument and a diagonal argument there exists some subsequence $\{\varphi_{m_i}\}_i \subset \{\varphi_m\}_m$ and $\varphi \in C^{1,\frac{\lambda}{2}}(\bar{A}_k)$ such that

$$\varphi_{m_i} \to \varphi \quad C^{1,\frac{\lambda}{2}}(\bar{A}_k).$$

So we can write

$$\varphi_{m_i} \to \varphi \qquad C^{1,\frac{\lambda}{2}}_{loc}(\bar{B}_1\backslash\{0\}).$$

Suppose t_m converges to some $t \in [0,\infty]$ and by passing the limit we see that when $t \in [0,\infty)$, φ solves

$$-\Delta\varphi + \frac{p}{(t|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\varphi}{|x|^2} = 0 \quad \text{in} \;\; B_1\backslash\{0\}$$

with $\varphi = 0$ on ∂B_1. When $t \to \infty$, we get

$$-\Delta\varphi = 0 \quad \text{ in } B_1\backslash\{0\}$$

with $\varphi = 0$ on ∂B_1. Using the completeness of $\mathbb{R}^N$, we can pass to a subsequence and so x_m converges to some x_0 such that $|x_0|$ is bounded away from zero. We can now pass the limit in (34) to see that $|x_0|^{\sigma+1}|\nabla\varphi(x_0)| \geq \epsilon_0$ and this means $\varphi \neq 0$.

We need to show that φ is in X_1. So we need to show that φ belongs to X and it has no $k=0$ mode. We showed that for every fixed $0 < |x| < 1$, we have $|x|^{\sigma+1}|\nabla\varphi_{m_i}(x)| \leq 1$. So by passing to the limit, we get $|x|^{\sigma+1}|\nabla\varphi| \leq 1$. Thus by integration, we can show that we have $|x|^{\sigma}|\varphi(x)| \leq 1$, so we get

$$\sup_{0<|x|\leq 1}\left\{|x|^{\sigma}|\varphi(x)| + |x|^{\sigma+1}|\nabla\varphi(x)|\right\} < \infty.$$

This means that φ is in X. First note that since φ_m belong to X_1 for all $0 < r \leq 1$, we can write

$$\int_{|\theta|=1}\varphi_m(r\theta)d\theta = \sum_{k=1}^{\infty}a_k(r)\int_{|\theta|=1}\psi_{k,m}(\theta).1d\theta = 0. \tag{37}$$

This shows that φ_m has zero average over all the sphere for radius $0 < r \leq 1$. Now we use the convergence we obtained above, and we can write for all $0 < r \leq 1$ we have that $\theta \mapsto \varphi_m(r\theta)$ converges uniformly on S^{N-1} to $\theta \mapsto \varphi(r\theta)$. Thus for all $0 < r \leq 1$ we have

$$0 = \int_{|\theta|=1}\varphi_m(r\theta)d\theta \to \int_{|\theta|=1}\varphi(r\theta)d\theta.$$

So φ also has zero average over all the sphere for radius $0 < r \leq 1$ and hence φ is in X_1. This means that $\varphi \in X_1$ is nonzero. Thus for $t \in [0,\infty)$ we have a contradiction with the results from the previous lemmas. We now show that in case of $t \to \infty$ we also get a contradiction. We saw that when $t = \infty$, we have

$$\begin{cases} \Delta\varphi = 0 & \text{in } \ B_1\backslash\{0\}, \\ \varphi = 0 & \text{on } \ \partial B_1. \end{cases} \tag{38}$$

Then we write $\varphi = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$, and solving the equation gives us

$$a_k = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$$

where

$$\gamma_k^{\pm} = \frac{-(N-2)}{2} \pm \frac{\sqrt{(N-2)^2 + 4\lambda_k}}{2}.$$

In Lemma 2.4, we showed that $\gamma_k^- + \sigma < 0 < \gamma_k^+ + \sigma$ for all $k \geq 1$. Hence, to have φ in the required space, we must have $C_k = D_k = 0$ and so $\varphi = 0$. It is a contradiction with φ being nonzero.
Case 2: In this case, we assume there is some $\{x_m\}$ such that $|x_m| \to 0$ and $|x_m|^{\sigma+1}|\nabla\varphi_m(x_m)| \geq \epsilon_0 > 0$. Set $s_m = |x_m|$, so we have $s_m \to 0$. Define $z_m := s_m^{-1}x_m$ a sequence and note that by definition $|z_m| = 1$. Thus,

$$|z_m|^{\sigma+1}s_m^{\sigma+1}|\nabla\varphi_m(s_m z_m)| = s_m^{\sigma+1}|\nabla\varphi_m(s_m z_m)| \geq \epsilon_0 > 0. \tag{39}$$

Define $\zeta_m(z) := s_m^{\sigma}\varphi_m(s_m z)$ for $0 < |s_m z| < 1$. By (39)

$$|\nabla\zeta_m(z_m)| = s_m^{\sigma+1}|\nabla\varphi_m(s_m z_m)| \geq \epsilon_0 \tag{40}$$

and also we have the bounds

$$|z|^{\sigma}|\zeta_m(z)| \leq 1, \quad \text{and} \quad |z|^{\sigma+1}|\nabla\zeta_m(z)| \leq 1. \tag{41}$$

We can write

$$-\Delta\varphi_m(s_m z) + \frac{p}{\beta + t_m|s_m z|^{\xi-1}}\frac{s_m z.\nabla\varphi_m(s_m z)}{|s_m z|^2} = f^m(s_m z).$$

Using our definition, we can obtain $\Delta\zeta_m(z) = s_m^{\sigma+2}\Delta\zeta_m(s_m z)$, and $\nabla\zeta_m(z) = s_m^{\sigma+1}\zeta_m(s_m z)$. Thus a computation shows that

$$-\Delta\zeta_m(z) + \frac{p}{\beta + t_m s_m^{\xi-1}|z|^{\xi-1}}\frac{z.\nabla\zeta_m(z)}{|z|^2} = s_m^{\sigma+2}f^m(s_m z). \tag{42}$$

Note that by setting $g^m(z) := s_m^{\sigma+2}f^m(s_m z)$, we showed that

$$L^+_{t_m s_m^{\xi-1}}(\zeta_m(z)) = g^m(z) \quad \text{in} \quad B_{\frac{1}{s_m}} = E_m = \{z : 0 < |z| < \frac{1}{s_m}\}$$

with $\zeta_m = 0$ on ∂E_m. Thus

$$-\Delta\zeta_m(z) = g^m(z) - \frac{p}{\beta + t_m s_m^{\xi-1}|z|^{\xi-1}}\frac{z.\nabla\zeta_m(z)}{|z|^2} \quad \text{in} \quad E_m.$$

We define A_k and $\tilde{A}_k$ for $k \geq 2$ as

$$A_k = \Big\{z \in B_1 : \frac{1}{k} < |z| < k\Big\} \quad \text{and} \quad \tilde{A}_k = \Big\{z \in B_1 : \frac{1}{2k} < |z| < 2k\Big\}.$$

We have that $A_k \subset \tilde{A}_k$ and also $\tilde{A}_k \subset E_m$ for m big enough. Thus we obtained

$$-\Delta\zeta_m(z) = g^m(z) - \frac{p}{\beta + t_m s_m^{\xi-1}|z|^{\xi-1}} \frac{z.\nabla\zeta_m(z)}{|z|^2} \quad \text{in} \quad \tilde{A}_k.$$

First note that

$$\sup_{z\in\tilde{A}_k} |g^m| = \sup_{z\in\tilde{A}_k} s_m^{\sigma+2}|f^m(s_m z)| = \sup_{z\in\tilde{A}_k} \frac{|s_m|^{\sigma+2}|z|^{\sigma+2}|g^m|}{|z|^{\sigma+2}} \leq \big(\sup_{z\in\tilde{A}_k} \frac{\|f^m\|_Y}{|z|^{\sigma+2}}\big) \leq (2k)^{2+\sigma}\|f^m\|_Y. \tag{43}$$

Also we have

$$\begin{aligned}
&\left|\frac{p}{(t_m s_m^{\xi-1}|z|^{\xi-1}+\beta)} \frac{z\cdot\nabla\zeta_m}{|z|^2}\right| \leq \frac{p}{(t_m s_m^{\xi-1}|z|^{\xi-1}+\beta)} \frac{|z|^{\sigma+1}|\nabla\zeta_m|}{|z|^{\sigma+2}} \\
\leq &\frac{p}{(t_m s_m^{\xi-1}|z|^{\xi-1}+\beta)} \frac{|z|^{\sigma+1}|s_m|^{\sigma+1}|\nabla\varphi_m(s_m z)|}{|z|^{\sigma+2}} \leq \frac{p}{\beta}\frac{\|\varphi_m\|_X}{|z|^{\sigma+2}} = \frac{p}{\beta}\frac{1}{|z|^{\sigma+2}} \leq \frac{p}{\beta}(2k)^{2+\sigma} := \gamma_k.
\end{aligned}$$

Set $G = g^m(z) - \frac{p}{\beta+t_m s_m^{\xi-1}|z|^{\xi-1}} \frac{z.\nabla\zeta_m(z)}{|z|^2}$. By (43), we can show that G is bounded in $\tilde{A}_k$. Thus, there exists a $C > 0$ such that $\|G\|_{L^\infty(\tilde{A}_k)} \leq C$ and since $-\Delta\zeta_m = G$, we have $\|\Delta\zeta_m\|_{L^\infty(\tilde{A}_k)} \leq C$. By the elliptic regularity, we can say that for $0 < \lambda < 1$ there exists a positive constant C_1 such that

$$\|\zeta_m\|_{C^{1,\lambda}(\bar{A}_k)} \leq C_1.$$

Thus $\{\zeta_m\}_m$ is a bounded sequence in $C^{1,\lambda}(\bar{A}_k)$ for all $k \geq 2$. By standard compactness argument and a diagonal argument there exist some subsequence $\{\zeta_{m_i}\}_i \subset \{\zeta_m\}_m$ and $\zeta \in C^{1,\frac{\lambda}{2}}(\bar{A}_k)$ such that $\zeta_{m_i} \to \zeta$ in $C^{1,\frac{\lambda}{2}}(\bar{A}_k)$ so we get

$$\zeta_{m_i} \to \zeta \qquad C^{1,\frac{\lambda}{2}}_{loc}(\mathbb{R}^N\backslash\{0\}).$$

Suppose that $t_m s_m^{\xi-1}$ converges to some $t \in [0,\infty]$. By passing the limit in (42) we see that when $t < \infty$, ζ solves

$$-\Delta\zeta + \frac{p}{(t|z|^{\xi-1}+\beta)} \frac{z\cdot\nabla\zeta}{|z|^2} = 0 \quad \text{in} \;\; \mathbb{R}^N\backslash\{0\}$$

and in the case of $t = \infty$, we get

$$\Delta\zeta = 0 \;\; \text{in} \;\; \mathbb{R}^N\backslash\{0\}. \tag{44}$$

Using the completeness of $\mathbb{R}^N$, we can pass to a subsequence such that $z_m \to z_0$ with z_0 bounded away from zero and $|z_0| = 1$. We can now pass the limit in (40) to see that $|\nabla\zeta(z_0)| \geq \epsilon_0$ and this means that ζ is nonzero. We now need to show that ζ is in X_1. So we need to show that ζ belongs to X and it has no $k = 0$ mode. We showed that for every fixed $0 < |z| < 1$, we have $|z|^\sigma|\zeta_m(z)| \leq 1$, and $|z|^{\sigma+1}|\nabla\zeta_m(z)| \leq 1$, so by passing the limit in these two expressions, we get $|z|^{\sigma+1}|\nabla\zeta(x)| \leq 1$ and $|z|^\sigma|\zeta| \leq 1$ in $\mathbb{R}^N\backslash\{0\}$. Thus, ζ is in X. We know that ζ_m has no $k = 0$ mode so $\zeta_m \in X_1$. With a similar approach to case 1, we can show that since ζ_m is in X_1, it has zero average over the sphere with radius $0 < r \leq 1$. With the convergence that we have obtained, we find that ζ has also zero average over the sphere and so ζ is also in X_1. This shows that $\zeta \in X_1$ is nonzero and it satisfies $L_t(\zeta) = 0$ in $\mathbb{R}^N\backslash\{0\}$ which is a contradiction with the kernel results we obtained before.

When $t = \infty$, we have

$$\Delta\zeta = 0 \ \text{ in } \ \mathbb{R}^N \backslash \{0\}. \tag{45}$$

We can write $\zeta = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$. Solving the equation gives us

$$a_k = C_k(r^{\gamma_k^+}) + D_k(r^{\gamma_k^-})$$

where $\gamma_k^{\pm} = \frac{-(N-2)}{2} \pm \frac{\sqrt{(N-2)^2+4\lambda_k}}{2}$. In Lemma 2.4, we showed that $\gamma_k^+ + \sigma$ is positive and $\gamma_k^- + \sigma$ is negative and they are both nonzero and distinct. Hence, to have a_k in the required space, we should have $C_k = D_k = 0$. So $\zeta = 0$ and it is a contradiction with ζ being nonzero.

The results from case 1 and case 2 complete the proof of the claim we made in (33) which means we have

$$\sup_{0<|x|\le 1} |x|^{\sigma+1}|\varphi_m(x)| \to 0. \quad \square$$

As we mentioned before, this result gives us

$$\sup_{0<|x|\le 1} \{|x|^{\sigma}|\varphi_m|\} \to 0. \tag{46}$$

To show this, fix $0 < |x| < 1$ and let $\hat{x}$ be a point on the boundary of $B_1\backslash\{0\}$. Also, let $t_1 = \frac{1}{|x|}$ such that $g(t) = \varphi_m(tx)$. Then, $g'(t) = \nabla\varphi_m \cdot x$ and this gives us

$$|g(t_1) - g(1)| \le \int_1^{t_1} |g'(t)| \, dt.$$

Thus, we can write

$$|\varphi_m(\hat{x}) - \varphi_m(x)| = |\varphi_m(x)| \le \int_1^{t_1} |\nabla\varphi_m(tx)||x| \, dt. \tag{47}$$

We showed that $|\nabla\varphi_m(z)||z|^{\sigma+1} \le \epsilon$ for $0 < |z| \le 1$ and we can write

$$|\nabla\varphi_m(tx)|t^{\sigma+1}|x|^{\sigma+1} \le \epsilon. \tag{48}$$

By (47) and (48), we find that

$$|\varphi_m(x)| \le \int_1^{t_1} \frac{\epsilon|x|}{t^{\sigma+1}|x|^{\sigma+1}} \, dt = \frac{\epsilon}{|x|^{\sigma}} \int_1^{t_1} \frac{1}{t^{\sigma+1}} \, dt = \frac{\epsilon}{|x|^{\sigma}} \Big[\frac{t^{-\sigma}}{-\sigma}\Big]_1^{t_1} = \frac{\epsilon}{|x|^{\sigma}\sigma} \Big[\frac{-1}{t_1^{\sigma}} + 1\Big]$$

so we have

$$|x|^{\sigma}|\varphi_m(x)| \le \frac{\epsilon}{\sigma}[1 - \frac{1}{t_1^{\sigma}}] \le \frac{\epsilon}{\sigma}. \tag{49}$$

When ϵ goes to zero, we get that $|\varphi_m(tx)|t^{\sigma+1}|x|^{\sigma+1}$ approaches zero. By (49) we also find that $|x|^{\sigma}|\varphi_m(x)|$ goes to zero. Thus we have

$$\sup_{0<|x|\le 1}\{|x|^{\sigma+1}|\nabla\varphi_m(x)|+|x|^{\sigma}|\varphi_m(x)|\}\to 0.$$

This is a contradiction with $\|\varphi\|_X = 1$. □

• Set $\kappa = -p$ such that $\{\varphi_m\}_m \subset X_1$, $f^m \in Y_1$ satisfy

$$\begin{cases} -\Delta\varphi_m - \dfrac{p}{(t_m|x|^{\xi-1}+\beta)}\dfrac{x\cdot\nabla\varphi_m}{|x|^2} = f^m & \text{in} \quad B_1\backslash\{0\}, \\ \varphi_m = 0, & \text{on} \quad \partial B_1, \end{cases} \tag{50}$$

and we have the estimate

$$\|\varphi_m\|_X > C\|f^m\|_Y$$

such that

$$\epsilon_0 \le |x_m|^{\sigma+1}|\nabla\varphi_m(x_m)| \le 1. \tag{51}$$

We will skip the proof as it would be very similar to the other case. □

For the case $\kappa = +p$ one can use a continuation argument along with Theorem 2.5 and the studies that has been done on L_0 in [2] and [1] to show the following. At this part, we skip the details on the continuation argument in the case of $\kappa = -p$ and they will be studied later in Lemma 2.8 and Lemma 2.9.

Corollary 2.6. *There exists a positive constant C such that for all $f \in Y_1$ there is some $\varphi \in X_1$ such that $L_t^+(\varphi) = f$ in $B_1\backslash\{0\}$ with $\varphi = 0$ on ∂B_1 and $\|\varphi\|_X \le C\|f\|_Y$.*

To get the desired result on the full space X, we need to recombine it with the result for the $k = 0$ mode.

Lemma 2.7. *($k = 0$ mode for L_t^+) We are considering the case where $k = 0$ in (17) and for all positive t, $a_t(r)$ (with dependence on t) solves the equation. We are also assuming*

$$\sup_{0<r\le 1} r^{\sigma+2}|b(r)| \le 1 \tag{52}$$

thus we have

$$-a_t''(r) - \frac{(N-1)a_t'(r)}{r} + \frac{pa_t'(r)}{\beta r + tr^{\xi}} = b(r) \qquad 0<r<1 \tag{53}$$

with $a_t(1) = 0$. Note that $a_t(r)$ also satisfies

$$\sup_{0<|x|\le 1}\left\{r^{\sigma}|a_t(r)| + r^{\sigma+1}|a_t'(r)|\right\} \le C. \tag{54}$$

Proof. From (53), using the integrating factor $\mu_t(r)$, we get $-\frac{d}{d\tau}(\mu_t(\tau)a_t'(\tau)) = \mu_t(\tau)b(\tau)$. Noting that $\mu_t(1) = 1$, we can integrate both sides and we obtain

$$-\int_s^1 \frac{d}{d\tau}(\mu_t(\tau)a_t'(\tau))d\tau = \int_s^1 \mu_t(\tau)b(\tau)d\tau \Longrightarrow -(\mu_t(1)a_t'(1) - \mu_t(s)a_t'(s)) = \int_s^1 \mu_t(\tau)b(\tau)d\tau.$$

We can now deduce that

$$a'_t(s) = \frac{1}{\mu_t(s)}\Big(a'_t(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau\Big).$$

By integrating again with respect to s from r to 1, and considering $a_t(1) = 0$, we get

$$a_t(r) = -\int_r^1 \frac{1}{\mu_t(s)}\Big(a'_t(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau\Big)ds.$$

Set $a'_t(1) = -\int_{R_t}^1 \mu_t(\tau)b(\tau)d\tau$, where $R_t^{\xi-1}t = 1$ then we have

$$a'_t(r) = -(-1)\frac{1}{\mu_t(s)}\Big[-\int_{R_t}^1 \mu_t(\tau)b(\tau)d\tau + \int_r^1 \mu_t(\tau)b(\tau)d\tau\Big] = \frac{1}{\mu_t(s)}\int_r^{R_t} \mu_t(\tau)b(\tau)d\tau = -\frac{1}{\mu_t(r)}\int_{R_t}^r \mu_t(\tau)b(\tau)d\tau.$$

So we can write a_t as

$$a_t(r) = \int_r^1 \Big(\frac{1}{\mu_t(s)}\int_{R_t}^s \mu_t(\tau)b(\tau)d\tau\Big)ds, \quad 0 < r \le 1.$$

We now need to check that a_t satisfies the bounds independent of t for large t.

We should consider two cases: *(i)* $0 < r < R_t$, *(ii)* $R_t < r \le 1$.

For $r < R_t$, we have

$$\begin{aligned} r^{\sigma+1}|a'_t(r)| &\le r^{\sigma+2-N+\frac{p}{\beta}}\Big(\frac{tr^{\xi-1}+\beta}{t+\beta}\Big)^{-(\frac{p}{p-1})}\int_r^{R_t} \mu_t(\tau)|b(\tau)|\ d\tau \\ &\le r^{\sigma+2-N+\frac{p}{\beta}}\Big(\frac{tr^{\xi-1}+\beta}{t+\beta}\Big)^{-(\frac{p}{p-1})}\int_r^{R_t} \frac{\tau^{\sigma+2}}{\tau^{\sigma+2}}\mu_t(\tau)|b(\tau)|\ d\tau. \end{aligned}$$

Note that $-\sigma - 3 + N - \frac{p}{\beta} = -\xi$, so by (52), we can write

$$\begin{aligned} r^{\sigma+1}|a'_t(r)| &\le r^{\xi-1}\Big(\frac{t+\beta}{tr^{\xi-1}+\beta}\Big)^{(\frac{p}{p-1})}\int_r^{R_t} \tau^{N-1-\frac{p}{\beta}}\Big(\frac{t\tau^{\xi-1}+\beta}{t+\beta}\Big)^{(\frac{p}{p-1})}\frac{|b(\tau)|\tau^{\sigma+2}}{\tau^{\sigma+2}}d\tau \\ &\le \Big(\frac{1+\beta}{tr^{\xi-1}+\beta}\Big)^{(\frac{p}{p-1})}\Big(\frac{1-(\frac{R_t}{r})^{1-\xi}}{\xi-1}\Big) \le \Big(\frac{\beta+1}{\beta}\Big)^{\frac{p}{p-1}}\frac{1}{\xi-1}. \end{aligned}$$

Thus we proved

$$r^{\sigma+1}|a'_t(r)| \le \Big(\frac{\beta+1}{\beta}\Big)^{\frac{p}{p-1}}\frac{1}{\xi-1} \quad \text{for} \quad 0 < r < R. \tag{55}$$

For $r > R_t$, there exists a constant $c_q > 0$ such that we have $(a+b)^q \le c_q(a^q + b^q)$ for $q > 1$ and by using this inequality and noting that $\xi - \frac{p(\xi-1)}{p-1} < 0$, and $\xi > 1$, we can write

$$r^{\sigma+1}|a_t'(r)| \leq \frac{r^{\xi-1}}{(\beta+tr^{\xi-1})^{\frac{p}{p-1}}}\int_{R_t}^{r}\frac{(\beta+t\tau^{\xi-1})^{\frac{p}{p-1}}}{\tau^{\xi}}d\tau \leq \frac{C_1\beta^{\frac{p}{p-1}}r^{\xi-1}R_t^{1-\xi}}{(\beta+tr^{\xi-1})^{\frac{p}{p-1}}}+\frac{C_1(tr^{\xi-1})^{\frac{p}{p-1}}}{(\beta+tr^{\xi-1})^{\frac{p}{p-1}}}$$

where C_1 is a constant independent of t. Recall we have $tR_t^{\xi-1}=1$, thus

$$r^{\sigma+1}|a_t'(r)| \leq \frac{C_1\beta^{\frac{p}{p-1}}tr^{\xi-1}}{(\beta+tr^{\xi-1})^{\frac{p}{p-1}}}+\frac{C_1(tr^{\xi-1})^{\frac{p}{p-1}}}{(\beta+tr^{\xi-1})^{\frac{p}{p-1}}} \leq \frac{C_1\beta^{\frac{p}{p-1}}}{(tr^{\xi-1})^{\frac{p}{p-1}}}+C_1.$$

Since for $r \geq R_t$ we have $tr^{\sigma-1} \geq tR_t^{\sigma-1}=1$, we can deduce that

$$r^{\sigma+1}|a_t'(r)| \leq C_1\beta^{\frac{p}{p-1}}+C_1 = C_1(1+\beta^{\frac{p}{p-1}}) \quad \text{for} \quad r \geq R_t. \tag{56}$$

Combining (55) and (56) gives us

$$\sup_{0<r\leq 1} r^{\sigma+1}|a_t'(r)| \leq \max\left\{C_1(1+\beta^{\frac{p}{p-1}}),\ \left(\frac{\beta+1}{\beta}\right)^{\frac{p}{p-1}}\frac{1}{\xi-1}\right\}.$$

This shows that $a_t(r)$ satisfies the equation and is bounded independent of t. □

We will delay the proof of the mapping properties of L_t^- for now and we complete the proof of the main linear result assuming we have the L_t^- mapping properties.

Completion of the proof of Proposition 2.1. Here, assuming that we have the result from Lemma 2.9, we can combine it with Theorem 2.5 and Lemma 2.7 to complete the proof of Proposition 2.1. Let $f \in Y$ and $\varphi \in X$ satisfy (16) and we write $f(x)=f_0(r)+f_1(x)$, and $\varphi(x)=\varphi_0(r)+\varphi_1(x)$, where we have split off the $k=0$ mode from $\varphi_1 \in X_1$, and $f_1 \in Y_1$. Then by Theorem 2.5, we can write

$$\|\varphi\|_X \leq \|\varphi_0\|_X+\|\varphi_1\|_X \leq C|f_0\|_Y+C\|f_1\|_Y.$$

Hence, if we can show there is some constant $D>0$ (independent of f) such that $\|f_0\|_Y+\|f_1\|_Y \leq D\|f_0+f_1\|_Y$ then we would be done.

Given $f \in Y$, we have

$$f(x)=\sum_{k=0}^{\infty}b_k(r)\psi_k(\theta)=b_0(r)\psi_0(\theta)+\sum_{k=1}^{\infty}b_k(r)\psi_k(\theta)=b_0(r)+\sum_{k=1}^{\infty}b_k(r)\psi_k(\theta)=f_0+f_1$$

where $\psi_0(\theta)=1$ and $b_k(r)=\frac{1}{|S^{N-1}|}\int_{S^{N-1}}f(r,\theta)\psi_k(\theta)d\theta$.

Noting that

$$\int_{S^{N-1}=1}|f(r,\theta)|d\theta \leq \frac{\|f\|_Y}{r^{\sigma+2}}\left|S^{N-1}\right|,$$

we can obtain

$$\|f_0\|_Y=\sup_{0<|x|\leq 1}r^{\sigma+2}|b_0(r)|=\sup_{0<|x|\leq 1}r^{\sigma+2}\Big|\frac{1}{|S^{N-1}|}\int_{S^{N-1}}f(r\theta)d\theta\Big| \leq \sup_{0<|x|\leq 1}r^{\sigma+2}\frac{\|f\|_Y}{r^{\sigma+2}}=\|f\|_Y. \tag{57}$$

We know $f_1=f-f_0$ so we can write

$$\|f_1\|_Y = \|f - f_0\|_Y \le \|f\|_Y + \|f_0\|_Y \le 2\|f\|_Y. \tag{58}$$

By (57) and (58) we have

$$\|f_0\|_Y + \|f_1\|_Y \le D\|f\|_Y = D\|f_0 + f_1\|_Y$$

thus

$$\|\varphi\|_X \le D\|f_0 + f_1\|_Y = D\|f\|_Y \tag{59}$$

where D is a positive constant independent of f. Now we can go back to our previous notation to show that the main result we want in Proposition 2.1. We proved that if we fix (f, g) and set $F = f + g$ and $G = f - g$ and consider ζ_i to be a solution of the scalar problems

$$\begin{cases} -\Delta\zeta_1 - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\zeta_1 = F, & \text{in } B_1\backslash\{0\} \\ \zeta_1 = 0 & \text{on } \partial B_1, \end{cases} \quad \text{and}$$

$$\begin{cases} -\Delta\zeta_2 + p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\zeta_2 = G & \text{in } B_1\backslash\{0\}, \\ \zeta_2 = 0 & \text{on } \partial B_1, \end{cases}$$

we can define the operators

$$L_t^{\pm}(\zeta) = -\Delta\zeta \pm \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\zeta}{|x|^2}.$$

By (59) for some positive constants C_1 and C_2, we have the estimates

$$\|\zeta_1\|_X \le C_1\|F\|_Y \quad \text{and} \quad \|\zeta_2\|_X \le C_2\|G\|_Y.$$

We know φ, and ψ satisfy $\varphi + \psi = \zeta_1$ and $\varphi - \psi = \zeta_2$ From this we saw that

$$\varphi = \frac{\zeta_1 + \zeta_2}{2}, \quad \psi = \frac{\zeta_1 - \zeta_2}{2}. \tag{60}$$

So we get

$$\|\varphi\|_X = C_1\|\zeta_1 + \zeta_2\|_X \le C_1(\|\zeta_1\| + \|\zeta_2\|) \le C(\|F\|_Y + \|G\|_Y),$$

and

$$\|\psi\|_X = C_2\|\zeta_1 - \zeta_2\|_X \le C_2(\|\zeta_1\| + \|\zeta_2\|) \le C(\|F\|_Y + \|G\|_Y).$$

Since $F = f + g$, and $G = f - g$ we have

$$\|F\|_Y \le \|f\|_Y + \|g\|_Y \quad \text{and} \quad \|G\| \le \|f\|_Y + \|g\|_Y.$$

So there is some positive constant C such that for all $t \ge 0$ and for all f, and g in Y, there are some φ and ψ in X which satisfy

$$\begin{cases} -\Delta\varphi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi = f & \text{in } B_1\backslash\{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\varphi = g & \text{in } B_1\backslash\{0\}, \\ \varphi = \psi = 0 & \text{on } \partial B_1, \end{cases} \tag{61}$$

and we have the estimate

$$\|\varphi\|_X + \|\psi\|_X \le C\|f\|_Y + C\|g\|_Y.$$

This completes the main result of our linear theory, and Proposition 2.1. □

We now turn to the case of $\kappa = -p$. Consider (17) in the case of $\kappa = -p$ and $t = 0$ given by

$$-a_k''(r) - \frac{(N-1)a_k'}{r} + \frac{\lambda_k a_k(r)}{r^2} - \frac{p a_k'(r)}{\beta r} = b_k(r) \quad \text{for} \quad 0 < r < 1 \tag{62}$$

with $a_k(1) = 0$.

Lemma 2.8. *For all $k \ge 0$ there is some $C_k > 0$ such that for all functions $b(r)$ with $\sup_{0<r<1} r^{\sigma+2}|b(r)| \le 1$ there is some function $a(r)$ which satisfies (62) and $\sup_{0<r<1} r^{\sigma}|a(r)| \le C_k$.*

Proof. Here we assume $k \ge 1$ and $t = 0$. In Lemma 2.9 we will prove the result for the case where k is zero and t is positive. Our result will also hold true for $t = 0$. By assuming $k \ge 1$, we have an Euler type equation. We know that the fundamental set of solutions of homogeneous version of (62) play a crucial role. The solution of the homogeneous equation is given by $a_k(r) = C_1 r^{\gamma_k^+} + C_2 r^{\gamma_k^-}$ where

$$\gamma_k^{\pm} = \frac{-(N-2+\frac{p}{\beta})}{2} \pm \frac{\sqrt{(N-2+\frac{p}{\beta})^2 + 4\lambda_k}}{2}.$$

We can now use variation of parameters to write out the particular solution of (62) as

$$a_{k,p}(r) = u_1(r) r^{\gamma_k^+} + u_2(r) r^{\gamma_k^-}. \tag{63}$$

We know that

$$u_1'(r) r^{\gamma_k^+} + u_2'(r) r^{\gamma_k^-} = 0. \tag{64}$$

Thus we need to solve for u_1 and u_2. We compute $a_{k,p}$, $a_{k,p}'$ and $a_{k,p}''$ and plug in these values in (62) and we get

$$\begin{aligned} &u_1' r^{\gamma_k^+ - 1} + u_2' \gamma_k^- r^{\gamma_k^- - 1} + u_1 r^{\gamma_k^+ - 2}\left[{\gamma_k^+}^2 - \gamma_k^+ + \gamma_k^+\left[N - 1 - \frac{p}{\beta}\right] - \lambda_k\right] \\ &+ u_2 r^{\gamma_k^- - 2}\left[{\gamma_k^-}^2 - \gamma_k^- + \gamma_k^-\left[N - 1 - \frac{p}{\beta}\right] - \lambda_k\right] = -b_k(r). \end{aligned} \tag{65}$$

A computation shows that

$$\left[{\gamma_k^+}^2 - \gamma_k^+ + \gamma_k^+\left[N - 1 - \frac{p}{\beta}\right] - \lambda_k\right] = \left[{\gamma_k^-}^2 - \gamma_k^- + \gamma_k^-\left[N - 1 - \frac{p}{\beta}\right] - \lambda_k\right] = 0. \tag{66}$$

Thus we have

$$u_1' \gamma_k^+ r^{\gamma_k^+ - 1} + u_2' \gamma_k^- r^{\gamma_k^- - 1} = -b_k(r) \quad \text{and} \quad u_1' r^{\gamma_k^+} + u_2' r^{\gamma_k^-} = 0. \tag{67}$$

From these two equations, we can solve for u_1 and u_2 and we get:

$$u_1(r) = \int_{T_1}^{r} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^+ - 1}(\gamma_k^- - \gamma_k^+)} \quad \text{and} \quad u_2(r) = \int_{T_2}^{r} \frac{-b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)} \tag{68}$$

where C_1, C_2 and T_1 and T_2 are to be picked later. Thus the solution of the equation (62) can be written as:

$$a_k(r) = C_1 r^{\gamma_k^+} + C_2 r^{\gamma_k^-} + r^{\gamma_k^+} \int_{T_1}^{r} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^+ - 1}(\gamma_k^- - \gamma_k^+)} + r^{\gamma_k^-} \int_{T_2}^{r} \frac{-b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)}. \tag{69}$$

By the boundary condition, we need to have $a_k(1) = 0$. Thus we pick $T_1 = 1$, $C_2 = 0$, $T_2 = 0$ and C_1 such that we have

$$C_1 + \int_{0}^{1} \frac{-b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)} = 0 \Longrightarrow C_1 = \int_{0}^{1} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)}. \tag{70}$$

We first show that C_1 is well defined. Note that we showed that $\gamma_k^- + \sigma + 1 \leq 0$ and

$$\frac{|b_k(\tau)|}{\tau^{\gamma_k^- - 1}} \leq \frac{1}{\tau^{\gamma_k^- + \sigma + 1}}. \tag{71}$$

Thus we have

$$|C_1| \leq \left| \int_{0}^{1} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)} \right| \leq \frac{1}{|\gamma_k^- - \gamma_k^+|} \int_{0}^{1} \frac{d\tau}{\tau^{\sigma + \gamma_k^- + 1}} \leq \frac{1}{|\gamma_k^- - \gamma_k^+|} \frac{1}{|\sigma + \gamma_k^-|} \tag{72}$$

where this is bounded by some positive constant C. This shows that C_1 is bounded. So we found the equation of $a_k(r)$ as follows

$$a_k(r) = r^{\gamma_k^+} \int_{0}^{1} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)} + r^{\gamma_k^+} \int_{1}^{r} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^+ - 1}(\gamma_k^- - \gamma_k^+)} + r^{\gamma_k^-} \int_{0}^{r} \frac{-b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)}. \tag{73}$$

With this choice of parameters, $a_k(r)$ satisfies the equation (62) and the boundary condition. We now need to show that it satisfies the estimate as well. We have

$$r^\sigma |a_k(r)| \leq \left| \frac{r^\sigma r^{\gamma_k^+}}{\gamma_k^- - \gamma_k^+} \int_{0}^{1} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}} + \frac{r^\sigma r^{\gamma_k^+}}{\gamma_k^- - \gamma_k^+} \int_{r}^{1} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^+ - 1}} + \frac{r^\sigma r^{\gamma_k^-}}{\gamma_k^- - \gamma_k^+} \int_{0}^{r} \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}} \right|.$$

Noting that we have the estimate (71) and $\sigma + \gamma_k^+ > 0$, we see that there exists a positive constant C such that

$$\left| \frac{r^\sigma r^{\gamma_k^+}}{\gamma_k^- - \gamma_k^+} \int_{r}^{1} b_k(\tau) \frac{d\tau}{\tau^{\gamma_k^+ - 1}} \right| \leq r^{\gamma_k^+ + \sigma} \int_{r}^{1} |b_k(\tau)| \frac{d\tau}{\tau^{\gamma_k^+ - 1}} \leq \frac{r^{\gamma_k^+ + \sigma}}{|\sigma + \gamma_k^+|} \left(1 - \frac{1}{r^{\sigma + \gamma_k^+}} \right) \leq C r^{\gamma_k^+ + \sigma} \leq C. \tag{74}$$

We now need to examine the other two terms. Similarly, we should note that we have $\sup_{0<r<1} r^{\sigma+2} |b_k(r)| \leq 1$, and $\sigma + \gamma_k^+ > 0$. Thus we get

$$\left|\frac{r^{\gamma_k^+ +\sigma}}{\gamma_k^- - \gamma_k^+}\int_0^1 \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- -1}} + \frac{r^{\sigma} r^{\gamma_k^-}}{\gamma_k^- - \gamma_k^+}\int_0^r b_k(\tau)\frac{d\tau}{\tau^{\gamma_k^- -1}}\right| \leq \frac{1}{|\gamma_k^- - \gamma_k^+|}\left(\frac{r^{\gamma_k^+ +\sigma}}{|\sigma+\gamma_k^-|} + \frac{r^{\gamma_k^- +\sigma}}{|\sigma+\gamma_k^-| r^{\gamma_k^- +\sigma}}\right) \tag{75}$$
$$\leq r^{\gamma_k^+ +\sigma} + \leq C$$

The bounds we get from (74) and (75) give us the desired estimate on $a_k(r)$. □

Proof of Proposition 2.2. Recall we are trying to show there is some $C > 0$ such that for all $f \in Y$ and $t \geq 0$ there is some $\varphi \in X$ which satisfies

$$\begin{cases} L_t^{\pm}(\varphi) = f & \text{in } \ B_1\backslash\{0\}, \\ \varphi = 0 & \text{on } \ \partial B_1. \end{cases} \tag{76}$$

Moreover one has $\|\varphi\|_X \leq C\|f\|_Y$.

For the case $\kappa = +p$, the result has been proved in [1]. We are going to show the result is also true for the case $\kappa = -p$. First we prove the result on space X_1 and to get the desired result on the full space X, we will recombine it with the result for the $k = 0$ mode in Lemma 2.9. We fix some $0 < T < \infty$, and define the set of A to be all $t \in [0, T]$ such that there exists a $C_t > 0$ that for all $f \in Y_1$ there exists a $\varphi \in X_1$ satisfying (76) and the estimate

$$\|\varphi\|_X \leq C_T\|f\|_Y. \tag{77}$$

In Lemma 2.8, we showed that $0 \in A$, thus the set A is non-empty. We are going to show that A is closed and open.

First to show that is open, we let t_0 be in the set A, and we are going to show that for some small ϵ we have that $t = t_0 + \epsilon$ is also in the set A. This means that we need to show that there exists a $C_t > 0$ such that if we let $f \in Y_1$ then there exists $\varphi \in X_1$ such that

$$\begin{cases} L_t^{-}(\varphi) = f, & \text{in } \ B_1\backslash\{0\}, \\ \varphi = 0, & \text{on } \ \partial B_1, \end{cases} \tag{78}$$

and they satisfy the estimate. Since t_0 is in the set A, thus there exists a C_{t_0} such that for all $f \in Y_1$ there exists a φ_0 such that they satisfy

$$\begin{cases} L_t^{-}(\varphi_0) = f, & \text{in } \ B_1\backslash\{0\}, \\ \varphi_0 = 0, & \text{on } \ \partial B_1, \end{cases} \tag{79}$$

and the estimate

$$\|\varphi_0\|_X \leq C_{t_0}\|f\|_Y. \tag{80}$$

We look for a solution of the form $\varphi = \varphi_0 + \psi$ where ψ is unknown. We let $L_t^-(\varphi) = -\Delta\varphi + a_t\nabla\varphi$ where $a_t = \frac{-px}{(t|x|^{\xi-1}+\beta)|x|^2}$. Thus we want

$$-\Delta(\varphi_0+\psi) + a_t(\nabla\varphi_0 + \nabla\psi) = f$$
$$\Longleftrightarrow\ -\Delta\varphi_0 + a_{t_0}\nabla\varphi_0 - \Delta\psi + [a_{t_0+\epsilon} - a_{t_0}]\nabla\varphi_0 + a_{t_0+\epsilon}\nabla\psi = f.$$

By (79), we find that

$$
\begin{aligned}
& f - \Delta\psi + [a_{t_0+\epsilon} - a_{t_0}]\nabla\varphi_0 + a_{t_0+\epsilon}\nabla\psi = f \\
\Longleftrightarrow \ & -\Delta\psi + a_{t_0}\nabla\psi + [a_{t_0+\epsilon} - a_{t_0}]\nabla\varphi_0 + [a_{t_0+\epsilon} - a_{t_0}]\nabla\psi = 0 \\
\Longleftrightarrow \ & L^-_{t_0}(\psi) = [a_{t_0} - a_{t_0+\epsilon}]\nabla\varphi_0 + [a_{t_0} - a_{t_0+\epsilon}]\nabla\psi. \qquad (81)
\end{aligned}
$$

Thus, we need to find ψ such that it satisfies (81). Now let $f \in Y_1$ be nonzero and set $F := \frac{f}{\|f\|}$ so $\|F\|_Y = 1$. By noting that φ_0 satisfies (80) and $\|\psi\|_X \leq 1$, we want to show that φ is a solution of $L^-_t(\varphi) = f$, so we are going to apply a fixed point argument. Define the mapping $T_\epsilon(\psi) = \hat{\psi}$ such that

$$
L^-_{t_0}(\hat{\psi}) = [a_{t_0} - a_{t_0+\epsilon}]\nabla\varphi_0 + [a_{t_0} - a_{t_0+\epsilon}]\nabla\psi := f_1. \qquad (82)
$$

We are going to do a fixed point argument on $T_\epsilon : B_1 \to B_1$ where $B_1 = \{\psi \in X; \|\psi\|_X \leq 1\}$. We need to show that for some ϵ there exists a small $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$

- $T_\epsilon(B_1) \subset B_1$ (Into),
- there exists some $\gamma \in (0, 1)$ such that for all $\psi_1, \psi_2 \in B_1$ we have $\|T_\epsilon(\psi_2) - T_\epsilon(\psi_1)\|_X \leq \gamma\|\psi_2 - \psi_1\|_X$ (Contraction).

(I) Into. We have $L^-_{t_0}(\hat{\psi}) = f_1$. Let $f_1 := K + I$ where $K := [a_{t_0} - a_{t_0+\epsilon}]\nabla\varphi_0$ and $I := [a_{t_0} - a_{t_0+\epsilon}]\nabla\psi$. Thus we have

$$
\|f_1\| \leq \|K\|_Y + \|I\|_Y. \qquad (83)
$$

We can find

$$
\|K\|_Y \leq \sup_{0<|x|<1} |x|^{\sigma+2}[a_{t_0} - a_{t_0+\epsilon}]\nabla\varphi_0 \leq \sup_{0<|x|<1} |x|^{\sigma+1}|\nabla\varphi_0| \left| \frac{p}{((t_0+\epsilon)|x|^{\xi-1} + \beta)} - \frac{p}{((t_0)|x|^{\xi-1} + \beta)} \right|. \qquad (84)
$$

Since $\varphi_0 \in X$, we know that $\sup\limits_{0<|x|<1} |x|^{\sigma+1}|\nabla\varphi_0| \leq 1$, so we get

$$
\|K\|_Y \leq \sup_{0<|x|<1} |x|^{\sigma+1}|\nabla\varphi_0| \left| \frac{p\ \epsilon\ |x|^{\xi-1}}{[((t_0+\epsilon)|x|^{\xi-1} + \beta)]\,[((t_0)|x|^{\xi-1} + \beta)]} \right| \leq \epsilon \frac{p}{\beta^2} \qquad (85)
$$

and thus as ϵ goes to zero, $\|K\|_Y$ goes to zero. For $\|I\|_Y$, similarly we have

$$
\|I\|_Y \leq \sup_{0<|x|<1} |x|^{\sigma+2}[a_{t_0} - a_{t_0+\epsilon}]\nabla\psi \leq \sup_{0<|x|<1} |x|^{\sigma+1}|\nabla\psi| \left| \frac{p\ \epsilon\ |x|^{\xi-1}}{[((t_0+\epsilon)|x|^{\xi-1} + \beta)]\,[((t_0)|x|^{\xi-1} + \beta)]} \right|. \qquad (86)
$$

Since $\psi \in X$, we know that $\sup\limits_{0<|x|<1} |x|^{\sigma+1}|\nabla\psi| \leq 1$ so we get

$$
\|I\|_Y \leq \epsilon \frac{p}{\beta^2} \qquad (87)
$$

and thus as ϵ goes to zero $\|I\|_Y$ goes to zero. By (83), we can deduce that for ϵ small we have $T_\epsilon(B_1) \subset B_1$ and so T_ϵ is into.
(II) Contraction. We need to show that for some $\gamma \in (0, 1)$, we have

$$
\|T_\epsilon(\psi_2) - T_\epsilon(\psi_1)\|_X \leq \gamma\|\psi_2 - \psi_1\|_X. \qquad (88)
$$

We set the right hand side of (82) to be f_1 and we can write it as $f_1 := K(\varphi_0) + I(\psi)$ where $K := [a_{t_0} - a_{t_0+\epsilon}]\nabla\varphi_0$ and $I := [a_{t_0} - a_{t_0+\epsilon}]\nabla\psi$. Thus we can write $\|T_\epsilon(\psi_2) - T_\epsilon(\psi_1)\|_X$ as

$$\|K(\varphi_0) + I(\psi_2) - K(\varphi_0) - I(\psi_1)\|_Y = \|I(\psi_2) - I(\psi_1)\|_Y \leq \sup_{0<|x|<1} |x|^{\sigma+2}|\nabla\psi_2 - \nabla\psi_1|[a_{t_0} - a_{t_0+\epsilon}]$$

$$\leq \sup_{0<|x|<1} |x|^{\sigma+1}|\nabla(\psi_2 - \psi_1)| \left| \frac{p\ \epsilon\ |x|^{\xi-1}}{[((t_0+\epsilon)|x|^{\xi-1}+\beta)]\,[((t_0)|x|^{\xi-1}+\beta)]} \right|.$$

Similar to (86), we can find that

$$\|I(\psi_2) - I(\psi_1)\|_Y \leq \|\psi_2 - \psi_1\|_X \left| \frac{p\ \epsilon\ |x|^{\xi-1}}{[((t_0+\epsilon)|x|^{\xi-1}+\beta)]\,[((t_0)|x|^{\xi-1}+\beta)]} \right| \leq \epsilon\frac{p}{\beta^2}\|\psi_2 - \psi_1\|_X \tag{89}$$

For ϵ small, we get that there exists some $\gamma \in (0, 1)$ such that

$$\|I(\psi_2) - I(\psi_1)\|_Y \leq \gamma\|\psi_2 - \psi_1\|_X \tag{90}$$

which gives us

$$\|T_\epsilon(\psi_2) - T_\epsilon(\psi_1)\|_X \leq \gamma\|\psi_2 - \psi_1\|_X. \tag{91}$$

Thus, T_ϵ is a also a contraction. So we can apply Banach's Fixed point Theorem and thus there exists $\psi \in X_1$ such that it is the fixed point of T_ϵ. So we showed that there exists a constant C_t such that for $F \in Y_1$ where $\|F\| = 1$ there exists a φ in X_1 such that $L_t(\varphi) = F$. We show that there exists a $C_{t_1} > 0$ such that for all $f \in Y_1$ there exists $\varphi \in X_1$ such that they satisfy $L_t(\varphi) = f$ and the estimate. Thus, using the linearity of L_t, we can write

$$L_t(\varphi) = F = \frac{f}{\|f\|_Y} \Rightarrow \big(\|f\|_Y L_t^-(\varphi)\big) = L_t^-(\|f\|_Y\varphi) = f$$

Now we set $\varphi := \|f\|_Y\varphi$, so we have

$$L_t^-(\varphi) = f \quad \text{and} \quad \|\varphi\|_X \leq \|\|f\|_Y\varphi\|_X \leq \|\varphi\|_X\|f\|_Y \leq C_{t_1}\|f\|_Y. \tag{92}$$

Thus for all $f \in Y$ there exists some $\varphi \in X_1$ such that $L_t^-(\varphi) = f$ and $\|\varphi\|_X \leq C_{t_1}\|f\|$. This means that $t = t_0 + \epsilon$ is in the set A and thus A is open.

We now show that A is also closed. Let t_m be in A such that t_m converges to $t \in [0, T]$. The goal is to show that t is also in A. So we need to show that if we have $f \in Y_1$ we can find $\varphi \in X_1$ such that $L_t(\varphi) = f$ and they satisfy the estimate. Since $t_m \in A$ thus for all $f \in Y_1$ there exists $\varphi_m \in X_1$ such that $L_{t_m}(\varphi_m) = f$ and $\|\varphi_m\|_X \leq C_{t_m}\|f\|_Y$. From $L_{t_m}(\varphi_m) = f$, we get

$$-\Delta\varphi_m + a_t\nabla\varphi_m = f \tag{93}$$

where $a_t = \frac{-px}{(t|x|^{\xi-1}+\beta)|x|^2}$. Thus we have

$$-\Delta\varphi_m = -a_t\nabla\varphi_m + f := g_m. \tag{94}$$

We first assume C_{t_m} is bounded. Similar to before for $k \geq 2$ we define the two sets

$$A_k = \Big\{x \in B_1 : \frac{1}{k} < |x| < 1\Big\} \quad \text{and} \quad \tilde{A}_k = \Big\{x \in B_1 : \frac{1}{2k} < |x| < 1\Big\}.$$

such that $A_k \subset \tilde{A}_k$. With a similar approach as case $\kappa = -p$ in Theorem 2.5, we can show that g_m is bounded in $\tilde{A}_k$ and we have $\|\Delta\varphi_m\|_{L^\infty(\tilde{A}_k)} \leq C$. By elliptic regularity we get that $\|\varphi_m\|_{C^{1,\lambda}(\bar{A}_k)} \leq C_1$. Thus, we can use the compactness argument and the diagonal argument to deduce that there exists a subsequence $\{\varphi_{m_i}\}_i \subset \{\varphi_m\}_m$ and $\varphi \in C^{1,\frac{\lambda}{2}}(\bar{A}_k)$ such that $\varphi_{m_i} \to \varphi$ in $C^{1,\frac{\lambda}{2}}_{loc}(\bar{B}_1\backslash\{0\})$ and thus φ satisfies

$$-\Delta\varphi - \frac{p}{(t|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\varphi}{|x|^2} = f \quad \text{in} \quad B_1\backslash\{0\}. \tag{95}$$

Note that for fixed $0 < |x| < 1$ we have

$$\|\varphi\|_X = \sup_{0<|x|\leq 1}\left\{|x|^\sigma|\varphi| + |x|^{\sigma+1}|\nabla\varphi|\right\} \leq \lim_{m\to\infty}\sup_{0<|x|\leq 1}\left\{|x|^\sigma|\varphi_m| + |x|^{\sigma+1}|\nabla\varphi_m|\right\} \leq \lim_{m\to\infty}\|\varphi_m\|_X. \tag{96}$$

Thus, since $t_m \in A$, we get

$$\|\varphi\|_X \leq \lim_{m\to\infty}\|\varphi_m\|_X \leq C_{t_m}\|f\|_Y. \tag{97}$$

This gives us that φ satisfies $L_t^-(\varphi) = f$ and the estimate thus t is in A.

We now assume C_{t_m} is unbounded. Since by the assumption C_m is the smallest possible constant that we have the estimate for, we can say that we can find some $\varphi_m \in X_1$ and $f \in Y_1$ such that $L_{t_m}^-(\varphi_m) = f$ and $\|\varphi_m\|_X \leq C_{t_m}\|f\|_Y$. But for $(C_{t_m}-1)$, we do not have the estimate and thus we get $\|\varphi_m\|_X \geq (C_{t_m}-1)\|f\|_Y$. We first normalize and we get $\|\varphi_m\|_X = 1$ and $\|f\|_Y \longrightarrow 0$. We also have

$$-\Delta\varphi_m = -a_t\nabla\varphi_m + f := g_m \tag{98}$$

and with the same approach as above there exists a subsequence $\{\varphi_{m_i}\}_i \subset \{\varphi_m\}_m$ and $\varphi \in C^{1,\frac{\lambda}{2}}(\bar{A}_k)$ such that

$$-\Delta\varphi - \frac{p}{(t|x|^{\xi-1}+\beta)}\frac{x\cdot\nabla\varphi}{|x|^2} = 0 \quad \text{in } B_1\backslash\{0\}. \tag{99}$$

With a similar argument as in Theorem 2.5, we find that $\varphi \in X_1$ is nonzero and it in the kernel of L_t^- which a contradiction with our kernel results. Thus we can deduce that C_{t_m} should be bounded. We have shown that A is non-empty, open and closed meaning that for all $t \in (0,\infty)$ there exists C_t such that for all $f \in Y_1$ there exists $\varphi \in X_1$ such that

$$\begin{cases} L_t^-(\varphi) = f, & \text{in} \quad B_1\backslash\{0\}, \\ \varphi = 0, & \text{on} \quad \partial B_1, \end{cases} \tag{100}$$

and

$$\|\varphi\|_X \leq C_t\|f\|_Y. \tag{101}$$

So for all $0 < T < \infty$ there exists a $C^T \leq \infty$ such that $|C_t| \leq C^T$ for all $0 \leq t \leq T$. We should note that C_t can not approach infinity since if we assume that $C_t \longrightarrow \infty$ as $t \longrightarrow \infty$, we can use the same contradiction argument as in case $\kappa = -p$ in Theorem 2.5 (see [13] for the full details). This shows that C_t should be bounded and thus the proof is complete. □

Now to get the desired result on the full space X, we need to recombine it with the result for the $k = 0$ mode.

Lemma 2.9. *($k = 0$ mode for L_t^-) There is some positive constant C such that for all positive t and all $b(r)$ defined on $r \in (0,1)$ with infinite $\sup_{0<r<1} r^{\sigma+2}|b(r)|$, there exists some a_t which solves*

$$-a_t''(r) - \frac{(N-1)a_t'(r)}{r} - \frac{pa_t'(r)}{\beta r + tr^{\xi}} = b(r) \qquad 0 < r \le 1 \tag{102}$$

with $a_t(1) = 0$. Also, there exists $C > 0$ (independent of t) such that $a_t(r)$ satisfies

$$\sup_{0<r<1} r^{\sigma+1}|a_t'(r)| \le C \sup_{0<r<1} r^{\sigma+2}|b(r)|. \tag{103}$$

We are also assuming

$$\sup_{0<r\le 1} r^{\sigma+2}|b(r)| \le 1. \tag{104}$$

Proof. The same as before, we know the integrating factor associated with our ODE (102) is given by $\mu_t(r) = e^{P(r)}$ where

$$P(r) = \int_1^r \left(\frac{N-1}{\tau} + \frac{p}{\beta\tau + t\tau^{\xi}}\right)d\tau = (N-1)\ln r + \frac{p}{\beta(1-\xi)}\left(\ln\left(\frac{\beta r^{1-\xi} + t}{t+\beta}\right)\right).$$

By noting that $\beta(1-\xi) = 1-p$, the integrating factor is

$$\mu_t(r) = e^{(N-1)\ln r + \frac{p}{\beta(1-\xi)}\left(\ln\left(\frac{\beta r^{\xi-1}+t}{t+\beta}\right)\right)} = e^{\ln r^{(N-1)}}.e^{\ln\left(\frac{\beta r^{1-\xi}+t}{t+\beta}\right)^{(\frac{p}{\beta(1-\xi)})}} = r^{N-1+\frac{p}{\beta}}\left(\frac{tr^{\xi-1}+\beta}{t+\beta}\right)^{(\frac{p}{1-p})}.$$

Thus from (102), using the integrating factor, we get

$$-\frac{d}{d\tau}(\mu_t(\tau)a_t'(\tau)) = \mu_t(\tau)b(\tau).$$

By considering $\mu_t(1) = 1$, we can integrate both sides and obtain

$$-\int_s^1 \frac{d}{d\tau}(\mu_t(\tau)a_t'(\tau))d\tau = \int_s^1 \mu_t(\tau)b(\tau)d\tau \Longrightarrow -(\mu_t(1)a_t'(1) - \mu_t(s)a_t'(s)) = \int_s^1 \mu_t(\tau)b(\tau)d\tau.$$

Thus, we have

$$a_t'(s) = \frac{1}{\mu_t(s)}\left(a_t'(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau\right). \tag{105}$$

To obtain $a_t(r)$, we integrate (105) with respect to s from r to 1 and we consider $a_t(1) = 0$, so we deduce

$$a_t(r) = -\int_r^1 \frac{1}{\mu_t(s)}\left(a_t'(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau\right)ds.$$

We set $a_t'(1) = -\int_{R_t}^1 \mu_t(\tau)b(\tau)d\tau$ where $R_t^{\xi-1}t = 1$ then we have

$$a_t'(r) = -\frac{1}{\mu_t(r)} \int_{R_t}^{r} \mu_t(\tau) b(\tau) d\tau.$$

So we can write a_t as

$$a_t(r) = \int_r^1 \left(\frac{1}{\mu_t(s)} \int_{R_t}^{s} \mu_t(\tau) b(\tau) d\tau \right) ds \quad \text{for} \quad 0 < r \leq 1.$$

Thus by considering (104), we can find that

$$\begin{aligned} a_t(r) \leq & \int_r^1 \left(\frac{1}{\mu_t(s)} \int_{R_t}^{s} \frac{\mu_t(\tau)}{\tau^{\sigma+2}} \tau^{\sigma+2} |b(\tau)| d\tau \right) ds \\ \leq & \int_r^1 s^{1-N-\frac{p}{\beta}} \left(\frac{ts^{\xi-1}+\beta}{t+\beta} \right)^{-(\frac{p}{1-p})} \int_{R_t}^{s} \left(\frac{t\tau^{\xi-1}+\beta}{t+\beta} \right)^{(\frac{p}{1-p})} \tau^{-\xi+2\frac{p}{\beta}} \; \|b(r)\|_Y \\ \leq & C_t \|b(r)\|_Y. \end{aligned} \tag{106}$$

So we have shown that a_t satisfies the equation with an estimate, but C can possibly depend on t. Now assume the result is false and we suppose that there exists some positive t_m such that $C_{t_m} > M$. Also, a_{t_m}, and b_m satisfy (102) and

$$\|a_{t_m}\|_X > M \|b_m\|_Y.$$

By normalizing, we can assume that

$$\|b_m\|_Y = \sup_{0<r<1} \{ r^{\sigma+2} |b_m(r)| \} \to 0 \quad \text{and} \quad \|a_{t_m}\|_X = \sup_{0<r<1} \{ r^{\sigma} |a_{t_m}(r)| + r^{\sigma+1} |a_{t_m}'(r)| \} = 1. \tag{107}$$

We claim

$$\sup_{0<r<1} \{ r^{\sigma+1} |a_{t_m}'| \} \to 0. \tag{108}$$

Suppose there are some $0 < r_m < 1$ and $\epsilon > 0$ such that

$$r_m^{\sigma+1} |a_{t_m}(r_m)| \geq \epsilon. \tag{109}$$

We need to consider two cases:
(I) r_m is bounded away from zero,
(II) $r_m \searrow 0$.
In either case, recall that a_m satisfies

$$-a_{t_m}''(r) - \frac{(N-1)a_{t_m}'(r)}{r} - \frac{p a_{t_m}'(r)}{\beta r + t_m r^{\xi}} = b_m(r) \quad \text{for} \quad 0 < r < 1 \tag{110}$$

with $a_{t_m}(1) = 0$.
Case (I). Note that for all small positive $\mathbb{E}$, we have $\sup_{\mathbb{E}<r<1} |b_m(r)| \to 0$. Fix $\mathbb{E}_0 > 0$ small. We have that $|\frac{p a_{t_m}'(r)}{\beta r + t_m r^{\xi}}|$ is bounded by some positive C on $\mathbb{E}_0 < r < 1$. Using the regularity theorem of elliptic PDE, we can say for all $\mathbb{E}_0 > 0$, there exist some $0 < \lambda < 1$ and a positive constant C_1 such that

$$\|a_{t_m}\|_{C^{1,\lambda}(2\mathbb{E}_0<r<1)} < C_1.$$

So by the compactness argument and the diagonal argument, there exists some subsequence (without renaming) a_{t_m} such that $a_{t_m} \to a$ in $C^{1,\frac{\lambda}{2}}_{loc}(0,1]$. Now suppose t_m converges to some $t \in [0,\infty]$ and so we can pass to the limit in (110) to arrive at

$$-a''(r) - \frac{(N-1)a'(r)}{r} - \frac{pa'(r)}{\beta r + tr^{\xi}} = 0 \quad 0 < r < 1 \tag{111}$$

with $a(1) = 0$ when t is finite. Also, in the case where $t = \infty$ the equation is

$$-a''(r) - \frac{(N-1)a'(r)}{r} = 0 \quad 0 < r < 1$$

with $a(1) = 0$. Note that we can pass to a subsequence of r_m such that $r_m \to r_0 \in (0,1]$ and using the convergence we have, we get $r_0^{\sigma+1}|a'(r_0)| > \mathbb{E}$. The kernel results we obtained for $\kappa = -p$ in Lemma 2.4 show that this kernel is trivial and hence we have a contradiction.

Case (II). We first define

$$z_m(r) := r_m^{\sigma}\left\{a_{t_m}(r_m r) - a_{t_m}(r_m)\right\} \quad \text{for} \quad 0 < r < \frac{1}{r_m}$$

where $z_m(1) = 0$. We have

$$-a''_{t_m}(rr_m) - \frac{(N-1)a'_{t_m}(rr_m)}{rr_m} - \frac{pa'_{t_m}(rr_m)}{\beta rr_m + t_m(rr_m)^{\xi}} = b_m(rr_m) \quad 0 < rr_m < 1 \tag{112}$$

so we get

$$-r_m^{\sigma+2}a''_{t_m}(rr_m) - \frac{(N-1)r_m^{\sigma+1}a'_{t_m}(rr_m)}{r} - \frac{pr_m^{\sigma+1}a'_{t_m}(rr_m)}{\beta r + t_m(r)^{\xi}(r_m)^{\xi-1}} = r_m^{\sigma+2}b_m(rr_m).$$

Thus

$$-z''_m(r) - \frac{(N-1)z'_m(r)}{r} - \frac{pz'_m(r)}{\beta r + r^{\xi}t_m(r_m)^{\xi-1}} = r_m^{\sigma+2}b_m(rr_m) =: g_m(r) \quad 0 < r < \frac{1}{r_m} \tag{113}$$

where $z_m(1) = 0$. Also note that we have

$$\sup_{0<r<\infty} r^{\sigma+2}|g_m(r)| = \sup_{0<rr_m<1} (rr_m)^{\sigma+2}|b_m(r_m r)| \to 0.$$

Considering the boundary condition and (109), we can write

$$z_m(1) = 0, \quad |z'_m(1)| = r_m^{\sigma+1}|a'_{t_m}(r_m)| \geq \epsilon, \quad \text{and} \quad r^{\sigma+1}|z'_m(r)| = (rr_m)^{\sigma+1}|a_{t_m}(r_m r)| \leq 1. \tag{114}$$

Since $\left|\frac{pz'_m(r)}{\beta r + r^{\xi}t_m(r_m)^{\xi-1}}\right|$ is bounded, we can apply a similar argument as before and so we have $\{z_m\}_m$ is a bounded sequence is $C^{1,\lambda}$ away from the origin and infinity, thus there exists a subsequence (without renaming) $\{z_m\}_m$ such that z_m converges to some z in $C^{1,\frac{\lambda}{2}}$. Now we can pass to the limit in (113). We need to consider three cases depending on the limiting behavior of $t_m(r_m)^{\xi-1}$:

(i) $t_m(r_m)^{\xi-1} \to 0$,

(ii) $t_m(r_m)^{\xi-1} \to t \in (0,\infty)$,

(iii) $t_m(r_m)^{\xi-1} \to \infty$.

Case (i): Assuming that $t_m(r_m)^{\xi-1}$ goes to zero, by passing the limit in equation (113), we get

$$-z''(r) - \frac{(N-1)z'(r)}{r} - \frac{pz'(r)}{\beta r} = 0 \quad 0 < r < \infty$$

where $z(1) = 0$. Since away from the origin and infinity we have the $C^{1,\lambda}$ convergence, we can pass the limit in (114). Thus we have

$$|z'(1)| \geq \epsilon, \tag{115}$$

and

$$r^{\sigma+1}|z'(r)| \leq 1.$$

Since $z(r)$ is in the required space, we can use the kernel results we obtained in the case of $k = 0$, and $t = 0$. So we should have $z(r) = 0$ on $0 < r < \infty$. But this is a contradiction with (115).
Case (ii): In this case, we can use a similar approach as case (i) and by passing the limit in equation (113), we get

$$-z''(r) - \frac{(N-1)z'(r)}{r} - \frac{pz'(r)}{\beta r + t} = 0 \quad 0 < r < \infty$$

and so using the convergence we have obtained, we can pass to the limit in (114). Thus we have

$$|z'(1)| \geq \epsilon, \tag{116}$$

and

$$r^{\sigma+1}|z'(r)| \leq 1.$$

We again use the Lemma 2.4 in the case $t \neq 0$ where similarly gives us the result of z being zero. Thus have the same contradiction as case (i).
Case (iii): When $t_m(r_m)^{\xi-1}$ approaches infinity, we get

$$-z''(r) - \frac{(N-1)z'(r)}{r} = 0 \quad 0 < r < \infty$$

where $z(1) = 0$. We can use the result from Lemma 2.4 in the case of t approaching infinity which stated we should have $z(r) = 0$. Thus again we have a contradiction with (116).

The contradictions from these 3 cases prove that we have the estimate (106) where C is independent of t. This can complete the result and thus we have shown that a_t satisfies the equation and we have the estimate independent of t. □

3. The non-linear theory

The goal in this chapter is to show that our nonlinear mapping $T_t(\varphi, \psi)$ is into and a contraction. Thus by applying Banach's Fixed Point Theorem we can obtain its fixed point and complete the proof of Theorem 1.1. Before that, we need to show the estimates and the asymptotic results that we will need for the main proof.

3.1. Estimates

Lemma 3.1. *Suppose $1 < p \le 2$. Then there exists some positive C such that for $x, y, z \in \mathbb{R}^N$*

$$\big||x+y|^p - |x|^p - p|x|^{p-2}x\cdot y\big| \le C|y|^p. \tag{117}$$

$$\big||x+y|^p - p|x|^{p-2}x\cdot y - |x+z|^p + p|x|^{p-2}x\cdot z\big| \le C\left(|y|^{p-1}+|z|^{p-1}\right)|y-z|. \tag{118}$$

$$||x+y|^p - |x+z|^p| \le C\left(|x|^{p-1}+|y|^{p-1}+|z|^{p-1}\right)|y-z|. \tag{119}$$

Proof. We skip the proof here. The proofs with all the details are available at [13]. □

3.2. Asymptotics

To prove Theorem 1.1, we will also need some asymptotics of w_t. Note that we had

$$w_t(r) = \int_r^1 \frac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}, \quad \text{and} \quad w_t'(r) = 0 - \frac{1}{(tr^\xi + \beta r)^{\frac{1}{p-1}}} = \frac{-1}{(tr^\xi + \beta r)^{\frac{1}{p-1}}}.$$

This means that for $0 < r < 1$, we have $|w_t'(r)| \le \min\{\frac{C_\beta}{r^{\frac{1}{p-1}}}, \frac{1}{t^{\frac{1}{p-1}} r^{N-1}}\}$ where $C_\beta = \frac{1}{\beta^{\frac{1}{p-1}}}$. This gives us $r^{\sigma+1}|w_t'(r)| \le \min\{C_\beta, \frac{1}{t^{\frac{1}{p-1}} r^{N-2-\sigma}}\}$. We should also note that for any positive t, we have $\lim_{r\to 0} r^{\sigma+1} w_t'(r) = -C_\beta$. To show this, noting that $\sigma + 1 = \frac{1}{p-1}$ we can write that

$$\lim_{r\to 0} r^{\sigma+1} w_t'(r) = \lim_{r\to 0} \frac{-r^{\sigma+1}}{(tr^\xi + \beta r)^{\frac{1}{p-1}}} = \lim_{r\to 0} \frac{-r^{\sigma+1}}{r^{\frac{1}{p-1}}(tr^{\xi-1} + \beta)^{\frac{1}{p-1}}} = \lim_{r\to 0} \frac{-1}{(tr^{\xi-1} + \beta)^{\frac{1}{p-1}}} = \frac{-1}{\beta^{\frac{1}{p-1}}} = -C_\beta.$$

Also, away from zero, we can see that $w_t(r)$ and $w_t'(r)$ converge uniformly to zero.

3.3. The fixed point argument

We defined the nonlinear mapping $T_t(\varphi, \psi) = (\hat{\varphi}, \hat{\psi})$ via

$$\begin{cases} -\Delta\hat{\varphi} - p|\nabla w|^{p-2}\nabla w\cdot\nabla\hat{\psi} = \kappa_1(x)|\nabla w + \nabla\psi|^p + I(\psi) & \text{in } B_1\backslash\{0\}, \\ -\Delta\hat{\psi} - p|\nabla w|^{p-2}\nabla w\cdot\nabla\hat{\varphi} = \kappa_2(x)|\nabla w + \nabla\varphi|^p + I(\varphi) & \text{in } B_1\backslash\{0\}, \\ \varphi = \psi = 0 & \text{on } \partial B_1, \end{cases} \tag{120}$$

where

$$I(\zeta) = |\nabla w + \nabla\zeta|^p - |\nabla w|^p - p|\nabla w|^{p-2}\nabla w\cdot\nabla\zeta.$$

So for $R > 0$, we define the space $\mathcal{F}_R$ as

$$\mathcal{F}_R := \{(\varphi, \psi) \in X \times X : \|\varphi\|_X, \|\psi\|_X \le R\}.$$

We will show that T_t is a contraction on $\mathcal{F}_R$ for suitable R and t. Also, note that on this space we have

$$\|(\varphi, \psi)\|_{X\times X} := \|\varphi\|_X + \|\psi\|_X. \tag{121}$$

Lemma 3.2. *To show that our mapping T_t is into, we need to show that for all $(\varphi,\psi) \in \mathcal{F}_R$, we have $T_t(\varphi,\psi) \in \mathcal{F}_R$. We know that by (121), we have*

$$\|T_t(\varphi,\psi)\|_{X\times X} = \|(\hat{\varphi},\hat{\psi})\|_{X\times X} = \|\hat{\varphi}\|_X + \|\hat{\psi}\|_X.$$

First we set the right hand side of (120) to be $H_1(\psi)$ and $H_2(\varphi)$ as $H_1(\psi) = J_t(\psi) + Q_t(\psi)$ and $H_2(\varphi) = I_t(\varphi) + K_t(\varphi)$ where

$$J_t(\psi) := |\nabla w_t + \nabla\psi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t \nabla\psi, \quad \text{and} \quad Q_t(\psi) := \kappa_1|\nabla w_t + \nabla\psi|^p,$$
$$I_t(\varphi) := |\nabla w_t + \nabla\varphi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t \nabla\varphi, \quad \text{and} \quad K_t(\varphi) := \kappa_2|\nabla w_t + \nabla\varphi|^p.$$

To get the result we need, we should prove two statements for both $H_1(\psi)$ and $H_2(\varphi)$. First we state them for $H_1(\varphi)$.

1. *There is some positive constant C such that for $R \in (0,1)$, $0 < \delta < 1$, $t > 1$, and $\psi \in B_R \subset X$ one has*

$$\|\kappa_1|\nabla w_t + \nabla\psi|^p\|_Y \le C\left(R^p + \sup_{|z|<\delta}|k_1(z)| + \frac{1}{t^{\frac{p}{p-1}}\delta^{(N-1)p-\sigma-2}}\right).$$

2. *There is some positive constant C such that for all $t > 1$, $0 < R < 1$ and $\psi \in B_R$ one has*

$$\||\nabla w_t + \nabla\psi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t\nabla\psi\|_Y \le CR^p.$$

Proof. Fix R, and ψ as in the hypothesis and C would be a constant independent of these parameters. We have the estimates

$$(|x|^{\sigma+1}|\nabla\varphi(x)|)^p \le R^p \quad \text{and} \quad (|x|^{\sigma+1}|\nabla w_t(x)|)^p \le C. \tag{122}$$

First note that we have:

$$\|Q_t(\psi)\|_Y \le C \sup_{0<|x|\le 1} |x|^{\sigma+2}|x|^{-(\sigma+1)p}|\kappa_1|\big((|x|^{\sigma+1}|\nabla w_t|)^p + (|x|^{\sigma+1}|\nabla\psi|)^p\big).$$

We know that $\sigma + 2 - (\sigma+1)p = \frac{2-p}{p-1}(1-p) + 2 - p = 0$, thus we get

$$\|Q_t(\psi)\|_Y \le C \sup_{0<|x|\le 1} |\kappa_1|(|x|^{\sigma+1}|\nabla w_t|)^p + \sup_{0<|x|\le 1}|\kappa_1|(|x|^{\sigma+1}|\nabla\psi|)^p.$$

We saw that $w_t(r) = \int_r^1 \dfrac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}$, so

$$\|Q_t(\psi)\|_Y \le \sup_{0<|x|\le 1}|\kappa_1|\left(|x|^{\sigma+1}(t|x|^\xi + \beta|x|)^{\frac{-1}{p-1}}\right)^p + \sup_{0<|x|\le 1}|\kappa_1|(|x|^{\sigma+1}|\nabla\psi|)^p$$

and since $\sigma + 1 = \frac{1}{p-1}$, we get

$$\|Q_t(\psi)\|_Y \le \sup_{0<|x|\le 1}|\kappa_1|(t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{0<|x|\le 1}|\kappa_1|(|x|^{\sigma+1}|\nabla\psi|)^p.$$

Fix $0 < \delta \ll 1$. We have $\kappa_1(x)$, and $\kappa_2(x)$ are positive continuous functions such that $\kappa_1(0) = \kappa_2(0) = 0$. We write the $\sup_{0<|x|\le 1} |\kappa_1(x)|(t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{0<|x|\le 1} |\kappa_1(x)|(|x|^{\sigma+1}|\nabla\varphi|)^p$ as a supremum over B_δ and $\delta < |x| \le 1$ and this gives us

$$\|Q_t(\psi)\|_Y \le C\left(R^p + \sup_{B_\delta}|\kappa_2(x)| + \frac{1}{t^{\frac{p}{p-1}}|\delta|^{(\xi-1)\frac{p}{p-1}}}\right). \tag{123}$$

For $J_t(\psi)$, we apply the estimate (117) where we set $x = \nabla w_t$ and $y = \nabla\psi$. So we get

$$\|J_t(\psi)\|_Y \le \sup_{0<|x|\le 1} |x|^{\sigma+2}|\nabla\psi|^p \le \sup_{0<|x|\le 1} (|x|^{\sigma+1}|\nabla\psi|)^p \le CR^p. \tag{124}$$

By (123) and (124) we can deduce

$$\|H_1(\psi)\|_Y \le C\left(R^p + \sup_{B_\delta}|\kappa_1|(x) + \frac{1}{t^{\frac{p}{p-1}}|\delta|^{(\xi-1)\frac{p}{p-1}}}\right). \quad \square$$

We now prove similar statements for $H_2(\varphi)$.

1. *There is some positive constant C such that for $R \in (0,1)$, $0 < \delta < 1$, $t > 1$, and $\varphi \in B_R$ one has*

$$\|\kappa_2|\nabla w_t + \nabla\varphi|^p\|_Y \le C\left(R^p + \sup_{|z|<\delta}|\kappa_2(z)| + \frac{1}{t^{\frac{p}{p-1}}\delta^{(N-1)p-\sigma-2}}\right).$$

2. *There is some positive constant C such that for all $t > 1$, $0 < R < 1$ and $\varphi \in B_R$ one has*

$$\||\nabla w_t + \nabla\varphi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t\nabla\varphi\|_Y \le CR^p.$$

Proof. Fix R, and φ as in the hypothesis and C would be a constant independent of these parameters and we again use the estimates $(|x|^{\sigma+1}|\nabla\psi(x)|)^p \le R^p$, and $(|x|^{\sigma+1}|\nabla w_t(r)|)^p \le C$. First note that we have:

$$\|k_t(\varphi)\|_Y \le C \sup_{0<|x|\le 1} |x|^{\sigma+2}|x|^{-(\sigma+1)p}|\kappa_2|\big((|x|^{\sigma+1}|\nabla w_t|)^p + (|x|^{\sigma+1}|\nabla\varphi|)^p\big).$$

Since $\sigma + 1 = \frac{1}{p-1}$ and by definition of ω_t and $\sigma + 2 - (\sigma+1)p = 0$, thus we get:

$$\|k_t(\varphi)\|_Y \le \sup_{0<|x|\le 1} |\kappa_2|(t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{0<|x|\le 1} |\kappa_2|(|x|^{\sigma+1}|\nabla\varphi|)^p.$$

Similarly, fix $0 < \delta \ll 1$. We know that $\kappa_1(x), \kappa_2(x) > 0$ are continuous and $\kappa_1(0) = \kappa_2(0) = 0$. We write the $\sup_{0<|x|\le 1} |\kappa_2(x)|(t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{0<|x|\le 1} |\kappa_2(x)|(|x|^{\sigma+1}|\nabla\varphi|)^p$ as a supremum over B_δ and $\delta < |x| \le 1$. Noting that $(\xi - 1)\frac{p}{p-1} > 0$, we get

$$\|k_t(\varphi)\|_Y \le C\left(R^p + \sup_{B_\delta}|\kappa_2(x)| + \frac{1}{t^{\frac{p}{p-1}}|\delta|^{(\xi-1)\frac{p}{p-1}}}\right). \tag{125}$$

For $I_t(\varphi)$, we can apply (117), where we set $x = \nabla w_t$ and $y = \nabla\varphi$. So we get:

$$\|I_t(\varphi)\|_Y \le \sup_{0<|x|\le 1} |x|^{\sigma+2}|\nabla\varphi|^p \le \sup_{0<|x|\le 1} (|x|^{\sigma+1}|\nabla\varphi|)^p \le CR^p.$$

By (125), we obtain

$$\|H_2(\varphi)\|_Y \le C\bigg(R^p + \sup_{B_\delta}|\kappa_2|(x) + \frac{1}{t^{\frac{p}{p-1}}|\delta|^{(\xi-1)\frac{p}{p-1}}}\bigg). \quad \square$$

Lemma 3.3. *To show our mapping is a contraction, we are going to show that if we have $0 < R < 1,\ 0 < \delta < 1,\ t > 1$, and $\varphi_i, \psi_i \in B_R$ and we set $T_t(\varphi_i, \psi_i) = (\hat{\varphi}_i, \hat{\psi}_i)$, there exists some $C > 0$ such that*

$$\|T_t(\varphi_2, \psi_2) - T_t(\varphi_1, \psi_1)\|_{X\times X} \le C\big\{R^{p-1} + \sup_{B_\delta}\{|\kappa_1| + |\kappa_2|\} + \frac{1}{t\delta^{\xi-1}}\big\}\|(\varphi_2, \psi_2) - (\varphi_1, \psi_1)\|_{X\times X}.$$

Proof. First note that we have:

$$\begin{aligned}\|T_t(\varphi_2, \psi_2) - T_t(\varphi_1, \psi_1)\|_{X\times X} =& \|(\hat{\varphi}_2, \hat{\psi}_2) - (\hat{\varphi}_1, \hat{\psi}_1)\|_{X\times X} = \|[(\hat{\varphi}_2 - \hat{\varphi}_1), (\hat{\psi}_2 - \hat{\psi}_1)]\|_{X\times X} \\ &= \|\hat{\varphi}_1 - \hat{\varphi}_2\|_X + \|\hat{\psi}_1 - \hat{\psi}_2\|_X.\end{aligned}$$

We set the right hand sides of (120) to be $H_1(\psi)$ and $H_2(\varphi)$ where

$$\begin{aligned} H_1(\psi) &:= |\nabla w_t + \nabla\psi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t \nabla\psi + \kappa_1|\nabla w_t + \nabla\psi|^p, \\ H_2(\varphi) &:= |\nabla w_t + \nabla\varphi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t \nabla\varphi + \kappa_2|\nabla w_t + \nabla\varphi|^p. \end{aligned}$$

For $H_1(\psi)$ let us define I_1 and K_1 such that

$$H_1(\psi_2) - H_1(\psi_1) = I_1 + k_1$$

where

$$I_1(\psi_1, \psi_2) := |\nabla w_t + \nabla\psi_2|^p - p|\nabla w_t|^{p-2}\nabla w_t\nabla\psi_2 - |\nabla w_t + \nabla\psi_1|^p + p|\nabla w_t|^{p-2}\nabla w_t\nabla\psi_1 \tag{126}$$

and

$$k_1(\psi_1, \psi_2) := \kappa_1(x)(|\nabla w_t + \nabla\psi_2|^p - |\nabla w_t + \nabla\psi_1|^p). \tag{127}$$

Similarly, we write $H_2(\varphi)$ as

$$H_2(\varphi_2) - H_2(\varphi_1) = J_2 + Q_2$$

where

$$\begin{aligned} I_2(\varphi_1, \varphi_2) &:= |\nabla w_t + \nabla\varphi_2|^p - p|\nabla w_t|^{p-2}\nabla w_t\nabla\varphi_2 - |\nabla w_t + \nabla\varphi_1|^p + p|\nabla w_t|^{p-2}\nabla w_t\nabla\varphi_1 \\ Q_2(\varphi_1, \varphi_2) &:= \kappa_2(x)(|\nabla w_t + \nabla\varphi_2|^p - |\nabla w_t + \nabla\varphi_1|^p). \end{aligned}$$

We prove two claims for both H_1 and H_2 to show that T_t is a contraction. First we state and prove our claims for $H_1(\psi)$.

Claim 1. *There exists a positive constant $\hat{C}$ such that for $0 < R < 1$, $t \ge 0$ and $\varphi_i, \psi_i \in B_R$ we have*

$$\sup_{0<|x|\le 1} |x|^{\sigma+2}|k_1(\psi_1, \psi_2)| \le \hat{C}\ \big(\sup_{B_\delta}\ \kappa_1(x) + \frac{1}{t\delta^{\xi-1}} + 2R^{p-1}\big)\|(\varphi_2, \psi_2) - (\varphi_1, \psi_1)\|_{X\times X}.$$

Claim 2. *There exists a positive constant* $\tilde{C}$ *such that for* $0 < R < 1$, $t > 1$, *and* $\varphi_i, \psi_i \in B_R$ *we have:*

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|I_1(\psi_1,\psi_2)| \leq \tilde{C}2R^{p-1}\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}.$$

Proof of Claim 1. (We will skip some details of the following proof. The proof along with all the details is available at [13]) We use (5) and (127) and apply (119) where we take $x = \nabla w_t$, $y = \nabla\psi_2$ and $z = \nabla\psi_1$ Also we note that $(p-1)(\sigma+1) = 1$, thus we have:

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|k_1(\psi_1,\psi_2)| \leq \sup_{0<|x|\leq 1} |x|^{\sigma+2}\kappa_1(x)\big((|\nabla w_t + \nabla\psi_2|^p - |\nabla w_t + \nabla\psi_1|^p)\big)$$
$$\leq C_1 \sup_{0<|x|\leq 1}\left(\frac{\kappa_1(x)}{t|x|^{\xi-1}+\beta} + 2C_1R^{p-1}\kappa_1(x)\right)\|\psi_2-\psi_1\|_X.$$

Let $0 < \delta < 1$, so we find that:

$$\sup_{0<|x|\leq 1}\frac{\kappa_1(x)}{t|x|^{\xi-1}+\beta} \leq \sup_{0<|x|\leq \delta}\frac{\kappa_1(x)}{t|x|^{\xi-1}+\beta} + \sup_{\delta<|x|\leq 1}\frac{\kappa_1(x)}{t|x|^{\xi-1}+\beta} \leq C\sup_{B_\delta}\kappa_1(x) + \frac{C}{t\delta^{\xi-1}}. \tag{128}$$

Thus, there exists a positive constant $\hat{C}$ such that

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|k_1(\psi_1,\psi_2)| \leq \hat{C}\ \big(\sup_{B_\delta}\kappa_1(x) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}}\big)\|\psi_2-\psi_1\|_X.$$

We should note that

$$\|\psi_2-\psi_1\|_X \leq \|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X} = \|\psi_2-\psi_1\|_X + \|\varphi_2-\varphi_1\|_X \tag{129}$$

which means

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|k_1(\psi_1,\psi_2)| \leq \hat{C}\ \big(\sup_{B_\delta}\kappa_1(x) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}}\big)\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}. \quad \square \tag{130}$$

Proof of Claim 2. We apply (119) where we set $\nabla w_t = x,\ \nabla\psi_2 = y,\ \nabla\psi_1 = z$, thus we have

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|I_1(\psi_1,\psi_2)| \leq C \sup_{0<|x|\leq 1}\big((|x|^{(\sigma+1)}|\nabla\psi_2|^{p-1}) + (|x|^{(\sigma-1)}|\nabla\psi_1|^{p-1})\big)\|\psi_2-\psi_1\|_X$$
$$\leq \tilde{C}2R^{p-1}\|\psi_2-\psi_1\|_X.$$

Thus by (129), we can write

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|I_1(\psi_1,\psi_2)| \leq \tilde{C}2R^{p-1}\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}. \tag{131}$$

By (130) and (131), we can deduce that

$$\|\hat{\psi}_2-\hat{\psi}_1\|_X \leq \hat{C}\ \big(\sup_{B_\delta}\kappa_1(x) + \frac{1}{t\delta^{\xi-1}} + 2R^{p-1}\big)\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}. \quad \square \tag{132}$$

We now state similar claims for $H_2(\varphi)$.

Claim 3. *There exists a positive constant $\hat{C}$ such that for $0 < R < 1$, $t \geq 0$ and $\varphi_i, \psi_i \in B_R$*

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|Q_2(\varphi_1,\varphi_2)| \leq \hat{C}\ \big(\sup_{B_\delta} \kappa_2(x) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}}\big)\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}.$$

Claim 4. *There exists $\tilde{C} > 0$ such that for $0 < R < 1$, $t > 1$, and $\varphi_i, \psi_i \in B_R$ we have:*

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|J_2(\varphi_2,\varphi_2)| \leq \tilde{C}2R^{p-1}\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}.$$

Proof of Claim 3. With a similar proof to (130), we can apply (119) where we set $x = \nabla w_t$, $y = \nabla\psi_2$ and $z = \nabla\psi_1$ and we have:

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|Q_1(\varphi_1,\varphi_2)| \leq \sup_{0<|x|\leq 1} |x|^{\sigma+2}\kappa_2(x)\big((|\nabla w_t + \nabla\varphi_2|^p - |\nabla w_t + \nabla\varphi_1|^p)\big)$$
$$\leq C_1 \sup_{0<|x|\leq 1}\left(\frac{\kappa_2(x)}{t|x|^{\xi-1}+\beta} + 2C_1R^{p-1}\kappa_2(x)\right)\|\varphi_2-\varphi_1\|_X.$$

Similar to (128), by letting $0 < \delta < 1$, we have:

$$\sup_{0<|x|\leq 1}\frac{\kappa_2(x)}{t|x|^{\xi-1}+\beta} \leq C\sup_{B_\delta}\kappa_2(x) + \frac{C}{t\delta^{\xi-1}}.$$

Thus, by (129) there exists a positive constant $\hat{C}$ such that

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|Q_1(\varphi_1,\varphi_2)| \leq \hat{C}\ \big(\sup_{B_\delta} \kappa_2(x) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}}\big)\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}.\quad\square \tag{133}$$

Proof of Claim 4. Similar to the proof of (131), we can apply (118) where we take $\nabla w_t = x$, $\nabla\psi_2 = y$, $\nabla\psi_1 = z$, and thus we have

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|J_1(\varphi_1,\psi_2)| \leq C\sup_{0<|x|\leq 1}\big((|x|^{(\sigma+1)}|\nabla\varphi_2|^{p-1}) + (|x|^{(\sigma-1)}|\nabla\varphi_1|^{p-1})\big)\|\varphi_2-\varphi_1\|_X$$
$$\leq \tilde{C}2R^{p-1}\|\varphi_2-\varphi_1\|_X.$$

So by (129), we can write

$$\sup_{0<|x|\leq 1} |x|^{\sigma+2}|J_1(\varphi_1,\varphi_2)| \leq \tilde{C}2R^{p-1}\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}. \tag{134}$$

By (133) and (134), we can deduce that

$$\|\hat{\varphi}_2-\hat{\varphi}_1\|_X \leq \hat{C}\ \big(\sup_{B_\delta}\kappa_2(x) + \frac{1}{t\delta^{\xi-1}} + 2R^{p-1}\big)\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}. \tag{135}$$

So by (132) and (135), we have

$$\|T_t(\varphi_2,\psi_2)-T_t(\varphi_1,\psi_1)\|_{X\times X} = \|\hat{\varphi}_1-\hat{\varphi}_2\|_X + \|\hat{\psi}_1-\hat{\psi}_2\|_X$$
$$\leq C\big(\sup_{B_\delta}\{\kappa_1+\kappa_2\} + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}}\big)\|(\varphi_2,\psi_2)-(\varphi_1,\psi_1)\|_{X\times X}.\quad\square$$

3.4. Proof of the main theorem

Proof of Theorem 1.1. We can now complete the proof of our main theorem. Recall that we want to find some φ and ψ which satisfy (8) such that $u(x) = w_t(x) + \varphi(x)$, and $v(x) = w_t(x) + \psi(x)$ satisfy equation (4). We will show that the mapping T_t is a contraction on $\mathcal{F}_R$ for suitable $0 < R < 1$, and large t.
Into. Let $0 < R < 1$, $0 < \delta < 1$, $t > 1$, and $\varphi, \psi \in B_R$. Set $T_t(\varphi, \psi) = (\hat{\varphi}, \hat{\psi})$. Then by Lemma 3.2, there exists some $C > 0$ (independent of the parameters) such that

$$\|T_t(\psi, \varphi)\|_X = \|(\hat{\psi}, \hat{\varphi})\|_X \leq C\left(R^p + \sup_{B_\delta}\{|\kappa_1|(x) + |\kappa_2|(x)|\} + \frac{1}{t^{\frac{p}{p-1}}|\delta|^{(\xi-1)\frac{p}{p-1}}}\right).$$

Hence, for $(\hat{\varphi}, \hat{\psi})$ to be in $\mathcal{F}_R$, it is sufficient to have

$$C\left(R^p + \sup_{B_\delta}\{|\kappa_1|(x) + |\kappa_2|(x)|\} + \frac{1}{t^{\frac{p}{p-1}}|\delta|^{(\xi-1)\frac{p}{p-1}}}\right) \leq R. \tag{136}$$

Contraction. Let $0 < R < 1$, $0 < \delta < 1$, and $t > 1$, also $\varphi_i, \psi_i \in B_R$. Set $T_t(\varphi_i, \psi_i) = (\hat{\varphi}_i, \hat{\psi}_i)$. Then by Lemma 3.3 we have

$$\|(\hat{\varphi_2}, \hat{\psi_2}) - (\hat{\varphi_1}, \hat{\psi_1})\|_{X\times X} \leq C\big(\sup_{B_\delta}(\kappa_1 + \kappa_2) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}}\big)\|(\varphi_2, \psi_2) - (\varphi_1, \psi_1)\|_{X\times X}.$$

Hence, for T_t to be a contraction, it is sufficient to have

$$C\big(\sup_{B_\delta}(\kappa_1 + \kappa_2) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}}\big) \leq \frac{1}{2}. \tag{137}$$

We now choose the parameters. Note that we can satisfy both (136) and (137) by first taking $R > 0$ sufficiently small, similarly we take $\delta > 0$ sufficiently small and then we take t large enough. Thus we obtain that T_t is into and a contraction. We can now apply Banach's Fixed Point Theorem to find the fixed point (φ, ψ) of T_t on $\mathcal{F}_R$ and hence (φ, ψ) satisfies (8). Thus $u_t = u(x) = w_t(x)+\varphi(x)$, and $v_t = v(x) = w_t(x)+\psi(x)$ are solutions of (4) provided (φ, ψ) satisfies (8). We need to verify that we can obtain a nonzero positive solution. Note that we have

$$|x|^{\sigma+1}|\nabla u_t(x)| \geq r^{\sigma+1}|w_t'(r)| - |x|^{\sigma+1}|\nabla\varphi(x)| \geq r^{\sigma+1}|w_t'(r)| - R,$$
$$|x|^{\sigma+1}|\nabla v_t(x)| \geq r^{\sigma+1}|w_t'(r)| - |x|^{\sigma+1}|\nabla\psi(x)| \geq r^{\sigma+1}|w_t'(r)| - R$$

since $\varphi, \psi \in B_R$. Recall that for all positive t, we have $(\lim_{r\to 0} r^{\sigma+1}w_t'(r) = -C_\beta)$ and hence by taking $R > 0$ sufficiently small, we can see u_t, v_t are nonzero when for some small positive ϵ, we have $0 < |x| \leq \epsilon$. We want to show that both u_t and v_t are positive on $B_1\backslash\{0\}$. For all positive t, in the equation of (5), we can find some $z \in (0, 1)$ such that we have $ty^{\xi-1} < 1$ for any y that satisfies $0 < y < z$. Since we have $\frac{-1}{p-1} + 1 = \sigma$, we get:

$$w_t = \int_r^1 \frac{1}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}dy \geq \int_r^z \frac{1}{(y(ty^{\xi-1} + \beta))^{\frac{1}{p-1}}}dy \geq \int_r^z \frac{1}{y^{\frac{1}{p-1}}(1+\beta)^{\frac{1}{p-1}}} \geq C\big[-\sigma y^{-\sigma}\big]_r^z = C\big[\frac{1}{r^\sigma} - \frac{1}{z^\sigma}\big].$$

Thus there exists some positive constant $C = C_{(\beta,p)}$ such that for all $0 < r < z$, we have

$$r^\sigma w_t \geq C(1 - (\frac{r}{z})^\sigma)$$

and we know $u_t = w_t + \varphi$, and $v_t = w_t + \psi$ where φ, ψ are in X. Thus near the origin when r goes to zero, we see that u_t, v_t are positive. Now for $\epsilon < |x| < 1$, note that since $\kappa_i \geq 0$, we have $-\Delta u_t = (1+\kappa_1(x))|\nabla v|^p \geq 0$, and $-\Delta v = (1 + \kappa_2(x))|\nabla u|^p \geq 0$. Thus, by applying the maximum principle, we can say that u_t, v_t are both positive on $\epsilon < |x| < 1$. It verifies that we have nonzero positive solutions. □

Ethical approval

This research did not involve any studies with human participants or animals. Therefore, ethical approval was not required for this study.

Consent

The author agreed with the content of the paper and has given consent to the present submission.

Declaration of competing interest

The author declares no conflicts of interest.

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Data availability

No data were generated or analyzed during the course of this study, so data sharing is not applicable.

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